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A NOTE ON HYPERGEOMETRIC SERIES

By A. L. DIXON

[Received 10 May 1930]

THIS note is written to point out certain simplifications which are introduced into the theory of the hypergeometric series by the use of the notation of elliptic functions, and of the theory of their transformations. In particular, the relation between any three solutions of the hypergeometric equation is determined without much difficulty.

Supposing, to begin with, that $k^2<1$, $\alpha>0$, $\beta>0$, let us start from the integral

$$\int_0^X \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du = \tfrac{1}{2}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-k^2x)^{\gamma-1}\,dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}\left\{1+\frac{\alpha(1-\gamma)k^2}{\alpha+\beta}+\frac{\alpha(\alpha+1)(1-\gamma)(2-\gamma)}{1\,.\,2(\alpha+\beta)(\alpha+\beta+1)}k^4+\ldots\right\}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}F(1-\gamma,\alpha;\alpha+\beta;k^2).$$

Then, for general values of α and β, the Pochhammer double-circuit integrals

$$\int^{(P+,\,Q+,\,P-,\,Q-)}\mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du$$

where P, Q are any pair of quarter-periods, 0, K, $K'i$, $K+K'i$, ..., will be solutions of the hypergeometric equation; and, if we write $\alpha+\beta+\gamma+\delta=2$, all the twenty-four associated solutions are given by the interchanges of α, β, γ, δ, and corresponding changes in k^2 and K, $K'i$.

2. For convenience I take α, β, γ, δ all positive, so that the Pochhammer integrals can be replaced by line-integrals; this restriction can be removed afterwards by analytic continuation.

To ensure that the functions we consider shall be analytic and single-valued, we make the following restrictions:

(i) in the second integral the point $1/k$ must not be on the real axis between 0 and 1, so that for $F(1-\gamma,\alpha;\alpha+\beta;k^2)$, considered as a function of k^2, there must be a cut along the real axis from 1 to ∞;

(ii) the path of integration in $\int x^{\alpha-1}(1-x)^{\beta-1}(1-k^2x)^{\gamma-1}\,dx$ must not encircle the point $1/k$;

(iii) we may conveniently take the path of integration in

$$\int \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du,$$

as not going outside a quarter-period parallelogram.

In the matter of notation, u is usually associated with the modulus k, v with the modulus k', and w with some other modulus.

3. The twenty-four solutions can be obtained systematically, in six groups of four, by considering (i) the three obtained from the one written above, by adding a quarter-period K, $K'i$, $K+K'i$ to u in the integrand; for example, taking $\alpha>0$, $\beta>0$,

$$\left|\int_K^{2K} \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du\right| = \int_0^K k'^{2\beta+2\gamma-2}\,\mathrm{sn}^{2\beta-1}u\,\mathrm{cn}^{2\alpha-1}u\,\mathrm{dn}^{2\delta-1}u\,du$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}(1-k^2)^{\beta+\gamma-1}F(1-\delta,\beta;\alpha+\beta;k^2),$$

so that $(1-k^2)^{\beta+\gamma-1}F(1-\delta,\beta;\alpha+\beta;k^2) = F(1-\gamma,\alpha;\alpha+\beta;k^2)$, (1)

if we substitute $2K-u$ for u in the first integral; and (ii) by considering the six related values of k^2, and the corresponding quarter periods, viz.:

k^2	$1-k^2$	$1/k^2$	$1/(1-k^2)$	$(k^2-1)/k^2$	$k^2/(k^2-1)$
K	K'	$k(K-iK')$	$k'(K'-iK)$	kK'	$k'K$,

the quarter periods chosen being those for which $\mathrm{sn}=1$.

Thus, for example, Jacobi's imaginary transformation

$$\mathrm{sn}(iu,k) = i\,\mathrm{sc}(u,k'), \qquad \mathrm{cn}(iu,k) = \mathrm{nc}(u,k'), \qquad \mathrm{dn}(iu,k) = \mathrm{dc}(u,k'),$$

gives

$$\left|\int_0^{K'i} \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du\right| = \int_0^{K'} \mathrm{sn}^{2\alpha-1}v\,\mathrm{cn}^{2\delta-1}v\,\mathrm{dn}^{2\gamma-1}v\,dv$$

$$= \frac{\Gamma(\alpha)\Gamma(\delta)}{2\Gamma(\alpha+\delta)}F(1-\gamma,\alpha;\alpha+\delta;1-k^2), \quad (2)$$

and the transformation

$$k'\mathrm{sn}(u,k) = \mathrm{sd}(k'u, ik/k'), \qquad \mathrm{cn}(u,k) = \mathrm{cd}(k'u, ik/k'),$$

$$\mathrm{dn}(u,k) = \mathrm{nd}(k'u, ik/k'),$$

gives

$$\int_0^K \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du = k'^{-2\alpha}\int_0^{k'K} \mathrm{sn}^{2\alpha-1}w\,\mathrm{cn}^{2\beta-1}w\,\mathrm{dn}^{2\delta-1}w\,dw$$

or $F(1-\gamma,\alpha;\alpha+\beta;k^2) = (1-k^2)^{-\alpha}F\{1-\delta,\alpha;\alpha+\beta;k^2/(k^2-1)\}$. (3)

4. The relation between any three of the solutions can now be obtained by integrating around a quarter-period rectangle.

Cauchy's theorem tells us that

$$\int_0^K \mathrm{sn}^{2\alpha-1}z\,\mathrm{cn}^{2\beta-1}z\,\mathrm{dn}^{2\gamma-1}z\,dz + \int_K^{K+K'i} \mathrm{sn}^{2\alpha-1}z\,\mathrm{cn}^{2\beta-1}z\,\mathrm{dn}^{2\gamma-1}z\,dz$$

$$= \int_0^{K'i} \mathrm{sn}^{2\alpha-1}z\,\mathrm{cn}^{2\beta-1}z\,\mathrm{dn}^{2\gamma-1}z\,dz + \int_{K'i}^{K'i+K} \mathrm{sn}^{2\alpha-1}z\,\mathrm{cn}^{2\beta-1}z\,\mathrm{dn}^{2\gamma-1}z\,dz$$

which is transformed* into

$$\int_0^K \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du -$$

$$- e^{-\beta\pi i}k'^{2\beta+2\gamma-2}\int_0^{K'} \mathrm{sn}^{2\beta-1}v\,\mathrm{cn}^{2\gamma-1}v\,\mathrm{dn}^{2\delta-1}v\,dv$$

$$= e^{\alpha\pi i}\int_0^{K'} \mathrm{sn}^{2\alpha-1}v\,\mathrm{cn}^{2\delta-1}v\,\mathrm{dn}^{2\gamma-1}v\,dv -$$

$$- e^{(\alpha+\delta)\pi i}k^{2\alpha+2\beta-2}\int_0^{K} \mathrm{sn}^{2\delta-1}u\,\mathrm{cn}^{2\gamma-1}u\,\mathrm{dn}^{2\beta-1}u\,du$$

by the help of the transformations (1) and (2) already given.

Then, multiplying by $e^{(\beta+\gamma)\pi i}$, we get, on equating imaginary parts and using the identity $\alpha+\beta+\gamma+\delta=2$,

$$\sin(\beta+\gamma)\pi\int_0^K \mathrm{sn}^{2\alpha-1}u\,\mathrm{cn}^{2\beta-1}u\,\mathrm{dn}^{2\gamma-1}u\,du -$$

$$- k'^{2\beta+2\gamma-2}\sin\gamma\pi\int_0^{K'} \mathrm{sn}^{2\beta-1}v\,\mathrm{cn}^{2\gamma-1}v\,\mathrm{dn}^{2\gamma-1}v\,dv$$

$$= -\sin\delta\pi\int_0^{K'} \mathrm{sn}^{2\alpha-1}v\,\mathrm{cn}^{2\delta-1}v\,\mathrm{dn}^{2\gamma-1}v\,dv,$$

which may be written

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}F(1-\gamma,\alpha;\alpha+\beta;k^2)$$

$$= (1-k^2)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(1-\beta-\gamma)}{\Gamma(1-\gamma)}F(1-\delta,\beta;\beta+\gamma;1-k^2)+$$

$$+\frac{\Gamma(\alpha)\Gamma(1-\alpha-\delta)}{\Gamma(1-\delta)}F(1-\gamma,\alpha;\alpha+\delta;1-k^2).$$

* The determination of the trigonometrical factors outside the integrals can be conveniently left to the last stage. At the corner K, for example, $\mathrm{cn}\,u$ is transformed into $e^{-\frac{1}{2}\pi i}\,\mathrm{sn}\,v$; the rest of the integrand is continuous.

For this formula there will be a cut from $k^2=1$ to $k^2=\infty$, and another from $k^2=0$ to $k^2=-\infty$.

If we apply the transformation (3) to each term of this, and then write z for $(k^2-1)/k^2$, we get

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}F\left(1-\delta,\alpha;\alpha+\beta;\frac{1}{z}\right)$$

$$=(1-z)^{1-\alpha-\gamma}(-z)^{1-\delta}\frac{\Gamma(\beta)\Gamma(1-\beta-\gamma)}{\Gamma(1-\gamma)}F(1-\alpha,\beta;\beta+\gamma;z)+$$

$$+(-z)^{\alpha}\frac{\Gamma(\alpha)\Gamma(1-\alpha-\delta)}{\Gamma(1-\delta)}F(1-\beta,\alpha;\alpha+\delta;z)$$

where, now, there is a cut in the z-plane from 0 to ∞, so that, if z lies in the first or second quadrant, $-z=ze^{\pi i}$, and if z lies in the third or fourth, $-z=ze^{-\pi i}$.

By the help of the transformation (1) this takes Barnes's form*

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}F\left(1-\delta,\alpha;\alpha+\beta;\frac{1}{z}\right)$$

$$=(-z)^{1-\delta}\frac{\Gamma(\beta)\Gamma(1-\beta-\gamma)}{\Gamma(1-\gamma)}F(1-\delta,\gamma;\beta+\gamma;z)+$$

$$+(-z)^{\alpha}\frac{\Gamma(\alpha)\Gamma(1-\alpha-\delta)}{\Gamma(1-\delta).}F(1-\beta,\alpha;\alpha+\delta;z).$$

* Dr. E. W. Barnes, *Proc. London Math. Soc.*, Ser. 2, vol. vi (1908), p. 146; Whittaker and Watson, *Modern Analysis*, p. 289, where the final formula is given with four curious misprints.

THE DETERMINATION OF ALL UNIVERSAL TERNARY QUADRATIC FORMS

By A. OPPENHEIM

[Received 21 June 1930]

1. THE object of this note is to determine all indefinite ternary quadratic forms

$$f = ax^2+by^2+cz^2+2ryz+2szx+2txy, \tag{1.1}$$

of determinant d and integral coefficients $a, \ldots, 2t$, which represent all integers. Such a form is called *universal*. If r, s, t are all integers, f is said to be *classic*; otherwise *non-classic*.

THEOREM 1. *If f is classic and universal, then necessarily*

$$d \text{ is odd or double an odd integer}, \Omega = \mp 1, \tag{1.21}$$

and f is equivalent to

$$2yz-dx^2 \ (d \text{ odd}), \qquad 2yz+z^2-dx^2 \ (d \text{ even or odd}); \tag{1.22}$$

and each of these forms is universal.

Further, each represents zero properly.

THEOREM 2. *If q is non-classic and universal, then necessarily q is equivalent to the form*

$$yz-Dx^2 \quad (D = 4d, \ D \text{ integral}); \tag{1.31}$$

and this form represents all integers, including zero, properly.

Theorem 1 is implied by two theorems of Dickson* who proved (i) that (1.21) and (1.22) hold if f is classic, universal, and nul, and (ii) that any universal ternary form is necessarily nul.

My proof of Theorems 1 and 2 follows the lines of Dickson's proof of (ii). This proof depends essentially on the following important lemma due to Legendre.†

LEMMA. *Let the integers a, b, c be not all of like sign. Let a, b, c be relatively prime in pairs and let $-bc, -ca, -ab$ be quadratic residues of a, b, c respectively, so that integers A, B, C exist such that*

$$A^2 \equiv -bc \pmod{a}, \quad B^2 \equiv -ca \pmod{b}, \quad C^2 \equiv -ab \pmod{c}. \tag{1.41}$$

* Dickson (**1**) for (i); (**2**) for (ii).

† For an exposition see Dickson (**3**), Ch. VIII.

Then there exist integers x, y, z such that

$$ax^2+by^2+cz^2 = 0, \tag{1.42}$$

$$Az \equiv by \pmod{a}, \quad Bx \equiv cz \pmod{b}, \quad Cy \equiv ax \pmod{c}, \tag{1.43}$$

and ax, by, cz are relatively prime in pairs.

Before proceeding with the proof of Theorems 1 and 2, it may be well to point out that, by means of a deep theorem of Meyer* on the equivalence of indefinite ternary quadratic forms, we may state Theorem 1 very simply in terms of characters.

THEOREM 3. *Let f be an indefinite classic ternary quadratic form with reciprocal F and determinant d. The necessary and sufficient conditions that f be universal are* (1.21) *and*

$$(F|p) = (-\Omega|p) \tag{1.51}$$

for every prime p dividing $\Delta = d$.

Theorem 3 is due to Ross,† who, however, assumed (1.21). Naturally, in virtue of Theorem 1, this restriction may now be removed. Let us observe that the theorem of Ross affords an immediate practical test for determining whether a given classic indefinite ternary form is or is not universal.

2. Proof of Theorem I. Since f is universal it represents 1 and, we see, properly. We may therefore take $a = 1$, and, by the substitution of $x+ty+sz$ for x, a proper transformation,

$$f = x^2+\phi(y,z), \qquad \phi = by^2+2ryz+cz^2, \tag{2.1}$$

where, since the determinant is unaltered,

$$bc-r^2 = d = \Omega^2\Delta, \tag{2.2}$$

Ω and Δ being the invariants of f.

Let an odd prime p divide b, r, c. Then $f \equiv x^2 \pmod{p}$, whereas f, being universal, represents all residues modulo p. Let 4 divide b, $2r, c$. Then $f \equiv x^2 \pmod{4}$, whereas f represents all residues modulo 4. Hence

$$\text{the greatest common divisor of } b,\ 2r,\ c \text{ is 2 or 1.} \tag{2.3}$$

Our discussion falls into two parts :

$$(b, 2r, c) = 2, \qquad (b, r, c) = 2; \tag{2.31}$$

$$(b, 2r, c) = 1 \text{ or } 2, \qquad (b, r, c) = 1; \tag{2.32}$$

which are exhaustive.

* Meyer (**4**). † Ross (**5**).

2.4 Case (2.31). Plainly $d = 4\Delta$, $b = 2b'$, $c = 2c'$, $r = 2r'$, where $\Delta, \ldots, r'$ are integers. The form (b', r', c') is now properly primitive and has hessian Δ. We may assume therefore that b' is prime to 2Δ.

Now, from (2.1),

$$b'f = b'x^2 + 2v^2 + 2\Delta z^2, \qquad v = b'y + r'z.$$

In order that f should represent all odd residues modulo 8, x must be odd, i.e. $x^2 \equiv 1 \pmod 8$; consequently $v^2 + \Delta z^2$ must represent all residues modulo 4, whence $\Delta = 2m$, (m odd) and now

$$b'f = b'x^2 + 2v^2 + 4mz^2 \qquad (b', m \text{ odd}).$$

In order that f should represent all even residues modulo 16, we must have x even, say $x = 2X$; and

$$2b'X^2 + v^2 + 2mz^2$$

must represent all residues modulo 8. For the odd residues, v must be odd, i.e. $v^2 \equiv 1 \pmod 8$, and $b'X^2 + mz^2$ then represents all residues modulo 4. But this is impossible, since b' and m are both odd.

Hence, if f is universal, (2.31) cannot arise.

2.5 Case (2.32). According as $(b, 2r, c) = 1$ or 2, $\phi(y, z)$ is properly or improperly primitive. In the first case we may take b prime to $2d$; in the second case we may take $b = 2K$, K prime to $2d$. In the latter case d is odd.

Now, from (2.1),

$$bf = bx^2 + v^2 + dz^2, \qquad v = by + rz. \tag{2.51}$$

If $4 | d$, when necessarily b is odd, then

$$bf \equiv bx^2 + v^2 \pmod 4,$$

which has only three incongruent residues modulo 4.

Hence *d is odd or double an odd, and* $\Omega = \mp 1$, (2.52)

from (2.322), since the adjoint of (2.1) is

$$dX^2 + cY^2 - 2rYZ + bZ^2.$$

Observe that the integers b, 1, d are not all positive, since f is indefinite. We shall show that the remaining hypotheses of the lemma are satisfied if f is universal.

Let an odd prime p divide d. Then $-b$ is a quadratic residue of p. For if not, confining ourselves to such values of x, y, z as make f a multiple of p, we find from (2.51) that

$$bx^2 + v^2 \equiv 0, \qquad v^2 \equiv -bx^2, \qquad v \equiv x \equiv 0 \pmod p;$$

and then (2.51) yields $bf \equiv dz^2 \pmod{p^2}$, whereas f represents all multiples of p.

Hence $-b$ is a quadratic residue of p and so of p^λ for any $\lambda \geqslant 1$. Since d is not divisible by 4, $-b$ must be a quadratic residue of d. Further, from (2.2), $-d$ is a quadratic residue of b, since b is not divisible by 4.

The conditions of the lemma are satisfied, and consequently there exist integers α, β, γ such that

$$b\alpha^2+\beta^2+d\gamma^2 = 0, \qquad r\gamma \equiv \beta \pmod{b}, \tag{2.53}$$

with $b\alpha$, β, $d\gamma$ relatively prime in pairs.

From (2.532) there exists an integer δ such that

$$\beta = b\delta+r\gamma, \qquad (\delta \text{ prime to } \gamma). \tag{2.54}$$

Hence

$$\phi(\delta,\gamma) = -\alpha^2, \qquad (\delta \text{ prime to } \gamma).$$

We may therefore choose a proper transformation such that

$$\phi(y,z) = (-\alpha^2,\lambda,\mu), \qquad \lambda^2+\mu\alpha = r^2-bc = -d. \tag{2.55}$$

Hence f is equivalent to the form

$$x^2-\alpha^2y^2+2\lambda yz+\mu z^2$$

which, by substitution of x for $x-\alpha y$, is equivalent to

$$x(x+2\alpha y)+2\lambda yz+\mu z^2.$$

Since $(\alpha,d) = 1$, it follows from (2.552) that $(\lambda,\alpha) = 1$, so that integers r and s exist such that $\alpha s-\lambda r = 1$. The transformation $x_1 = \alpha x+\lambda z$, $z_1 = rx+sz$ is proper and yields (on dropping subscripts) the equivalent form

$$2xy+Ex^2+2Fxz+Gz^2.$$

Herein replace y by $y+kx-Fz$. We obtain

$$f \sim 2xy+(E+2k)x^2-Dz^2,$$

where $D = d$, since the determinant is unchanged. By choice of k, we have $E+2k = 0$ or 1 according as E is even or odd. And Theorem 1 follows at once. We observe that, if d is odd, the two forms of Theorem 1 are equivalent.

3. Proof of Theorem 2. Since q is non-classic and universal, it is plain that $f = 2q$ is improperly primitive and represents *all even* integers. Hence f represents 2 and, we see, properly. In the notation of (1.1) we may take $a = 2$; b and c are even and one at least of r, s, t is odd.

Our discussion falls into two cases: either (i) one of s and t, and

we may suppose t, is odd; or (ii) s and t are both even and consequently r is odd.

CASE (i), t *odd*. Let $A,\ldots, T$ be the co-factors of $a,\ldots, t$ in d.

Then $$-C \equiv 1 \pmod 4, \qquad -B \equiv s^2 \pmod 4, \tag{3.11}$$

$$d \text{ is even and } BC - R^2 = ad = 2d. \tag{3.12}$$

We now have

$$2f = u^2 + Cy^2 - 2Ryz + Bz^2, \qquad u = 2x + ty + sz. \tag{3.2}$$

Let an odd prime p divide C, R, B. Then $2f \equiv u^2 \pmod p$, whereas f represents all even residues modulo p. Since C is odd, $(C,\ R,\ B)$ and $(C,\ 2R,\ B)$ must therefore both be unity. Hence the form $\phi(y, z) = (C, -R, B)$ is properly primitive and we may suppose that C is prime to $2d$.

From (3.2) we obtain

$$2Cf = Cu^2 + v^2 + 2dz^2, \qquad v = Cy - Rz. \tag{3.21}$$

Let p be an odd prime divisor of d. As in § 2, we deduce that $-C$ is a quadratic residue of p and so of p^λ for any $\lambda \geqslant 1$.

Let $2d = 2^n\delta$ (δ odd and $n \geqslant 2$). If $n = 2$, then the last result and (3.111) show that $-C$ is a quadratic residue of $2d$.

If $n = 3$, so that $d \equiv 4 \pmod 8$, the residues of $2dz^2$ modulo 16 are 0 and 8. From (3.21) and the fact that f represents all even residues modulo 16, it follows that $Cu^2 + v^2$ and $Cu^2 + v^2 + 8$ must together represent all multiples of 4 modulo 16. Hence u and v are of like parity and the possible residues are

$$4C,\ 4,\ 4C+4,\quad C+9,\ 9C+1,\ C+1,\ 9C+9,$$

and those obtained by adding 8 to these. If $-C \equiv 5 \pmod 8$, the residue 8 is missing. Since $-C \equiv 1 \pmod 4$, we must have $-C \equiv 1 \pmod 8$.

If $n \geqslant 4$, we see that $Cu^2 + v^2$ must represent all multiples of 4 modulo 16, whence, as before, $-C \equiv 1 \pmod 8$.

In any case therefore $-C$ is a quadratic residue of 2^3 and so of 2^λ for any $\lambda \geqslant 1$.

We have thus proved that $-C$ is a quadratic residue of $2d$. Also, from (3.12), $-2d$ is a quadratic residue of C. Finally C and d are not both positive since f is indefinite. The lemma is applicable and there exist integers α, β, γ such that

$$C\alpha^2 + \beta^2 + 2d\gamma^2 = 0, \qquad -R\gamma \equiv \beta \pmod C, \tag{3.3}$$

and $C\alpha$, β, $2d\gamma$ are relatively prime in pairs.

Hence there exists an integer ϵ such that

$$\beta = C\epsilon - R\gamma, \qquad (\epsilon, \gamma) = 1. \tag{3.31}$$

Now

$$\begin{aligned} C(\alpha - t\epsilon - s\gamma) &= C\alpha - t(R\gamma + \beta) - Cs\gamma \\ &= C\alpha + aS\gamma - \beta t \equiv 0 \pmod 2, \end{aligned}$$

since $a = 2$ and $C\alpha, \beta, t$ are all odd. Hence $\alpha - t\epsilon - s\gamma = 2\rho$ (ρ integral). The integers ρ, ϵ, γ are such that $f(\rho, \epsilon, \gamma) = 0$.

We have proved consequently that with $y = \epsilon$, $z = \gamma$, the properly primitive form $Cy^2 - 2Ryz + Bz^2$ represents properly the number $-\alpha^2$, where $\alpha = t\epsilon + s\gamma + 2\rho$. By a proper transformation on y and z we have therefore

$$(C, -R, B) \sim (-\alpha^2, \lambda, \mu), \qquad \lambda^2 + \mu\alpha = -2d \tag{3.41}$$

and

$$2f = (2x + t'y + s'z)^2 + (-\alpha^2 y^2 + 2\lambda yz + \mu z^2). \tag{3.42}$$

Plainly t' is odd, for α, being prime to $2d$, is odd, while the coefficient of y^2 is even. But this coefficient is $t'^2 - \alpha^2$. Also we may suppose that s' is even. For, if s' be odd, replace y by $y + z$, and we obtain $s' + t'$, which is even, in place of s'. Next we may suppose that s' is zero. To do this, replace x by $x - \frac{1}{2}s'z$. Then μ and 2λ are both divisible by 4. We may take $t' = 1$ and $\alpha = 2\alpha' + 1$ and we obtain

$$2f \sim (2x + y)^2 - \alpha^2 y^2 + 4\lambda' yz + 4\mu' z^2,$$

where λ' and μ' are integers. Replace $x - \alpha' y$ by x. We obtain on division by 2 the form

$$2x(x + \alpha y) + 2\lambda' yz + 2\mu' z^2$$

equivalent to f. Since $(\alpha, 2d) = 1$, we have $(\lambda, \alpha) = 1$ and so $(\lambda', \alpha) = 1$. We proceed as in § 2, and we see that

$$f \sim 2yz - 2Dx^2, \qquad (D \text{ integral.})$$

Since $f = 2q$, Theorem 2 follows in case (i).

Case (ii). *s and t even, and r odd.* Replacing x by $x - \frac{1}{2}ty - \frac{1}{2}sz$, we may take

$$f = 2x^2 + by^2 + 2ryz + cz^2, \qquad 2(bc - r^2) = d, \tag{3.51}$$

where b and c are even and r is odd.

As before we see that $(b, r, c) = 1$ and $(b, 2r, c) = 2$. Thus the form (b, r, c) is improperly primitive, and we may therefore suppose that $b = 2K$ (K prime to $2d$). Also

$$bf = 2bx^2 + v^2 + mz^2, \qquad v = by + rz \tag{3.52}$$

where, from (3.512)

$$d = 2m, \qquad m \equiv 3 \pmod 4. \tag{3.53}$$

Arguments of the type already used show that $-2b$ is a quadratic residue of m. Further, $-m$ is a quadratic residue of K and so of $4K = 2b$, since $-m \equiv 1 \pmod 4$. Since b and m are not both positive, we can apply the lemma to (3.52). There exist integers α, β, γ such that

$$2b\alpha^2+\beta^2+m\gamma^2 = 0, \qquad \beta \equiv r\gamma \pmod{2b}, \tag{3.54}$$

and $2b\alpha$, β, $m\gamma$ are relatively prime in pairs. Hence there is an even integer δ with $\beta = r\gamma+b\delta$ and $(\delta, \gamma) = 1$. Hence for $y = \delta$, $z = \gamma$, the form (b, r, c) represents properly the number $-2\alpha^2$ and is therefore equivalent to the form $(-2\alpha^2, \lambda, 2\mu)$, $\lambda^2+4\mu\alpha^2 = m$. We now have

$$f \sim 2(x^2-\alpha^2y^2)+2\lambda yz+2\mu z^2,$$

and our reduction goes through precisely as before, since λ and 2α are prime to each other.

The proof of Theorem 2 is thus complete.

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A NOTE ON THE HYPERGEOMETRIC AND BESSEL'S EQUATIONS

By J. L. BURCHNALL AND T. W. CHAUNDY

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1. Introduction

In the theory of the hypergeometric differential equation

$$f(\delta)y = x^m g(\delta)y \tag{1}$$

where δ denotes xd/dx and f, g, are polynomial forms with constant coefficients, there is a theorem which, although elementary in character, does not appear to be so widely known as it deserves. It is of use in the classification of the equations (1) by their types of solution and may also be shown to have as consequences various well-known results in the theory of these equations. In its simplest form we may state the theorem thus:

(2) *If z is a solution of the differential equation*

$$(\delta-a)f(\delta)z = x^m g(\delta)z,$$

then

$$y = \prod_{s=0}^{r-1}(\delta-a-sm)z$$

is a solution of the differential equation

$$(\delta-a-rm)f(\delta)y = x^m g(\delta)y.$$

The proof is immediate for

$$\begin{aligned}(\delta-a-rm)f(\delta)y &= \prod_{s=0}^{r}(\delta-a-sm)f(\delta)z\\ &= \prod_{s=1}^{r}(\delta-a-sm)x^m g(\delta)z\\ &= x^m\prod_{s=0}^{r-1}(\delta-a-sm)g(\delta)z\\ &= x^m g(\delta)y.\end{aligned}$$

If everywhere we write x^{-1} for x, we convert (2) into the equivalent theorem:

(3) *If z is a solution of the differential equation*

$$f(\delta)z = x^m(\delta+b)g(\delta)z,$$

then

$$y = \prod_{s=0}^{r-1}(\delta+b+sm)z$$

is a solution of the differential equation

$$f(\delta)y = x^m(\delta+b+rm)g(\delta)y.$$

From (2) and (3) we deduce at once that

(4) *If z is a solution of the differential equation*

$$\prod_{s=1}^{p}(\delta-a_s)z = x^m\prod_{s=1}^{q}(\delta+b_s)z,$$

then

$$y = \prod_{s=1}^{p}\prod_{r=1}^{r_s-1}(\delta-a_s-rm)\prod_{s=1}^{q}\prod_{r=1}^{r'_s-1}(\delta+b_s+rm)z$$

is a solution of the differential equation

$$\prod_{s=1}^{p}(\delta-a_s-r_sm)y = x^m\prod_{s=1}^{q}(\delta+b_s+r'_sm)y.$$

More generally, though less usefully, we may assert that

(5) *If* $a_r, b_s \equiv a'_r, b'_s \pmod m \begin{pmatrix} r=1,\dots,p \\ s=1,\dots,q \end{pmatrix}$,

then solutions of the differential equations

$$\prod_{r=1}^{p}(\delta-a_r)y = x^m\prod_{s=1}^{q}(\delta+b_s)y,$$

$$\prod_{r=1}^{p}(\delta-a'_r)z = x^m\prod_{s=1}^{q}(\delta+b'_s)z,$$

are connected in pairs by relations of the form

$$u(\delta)y = v(\delta)z,$$

where u, v are polynomial forms with constant coefficients.

We employ also one other theorem rather different in character:

(6) *If $y(x)$ is a solution of the differential equation*

$$f(\delta)y = xg(\delta)y,$$

then $y(\omega x)$, where $\omega^m = 1$, is a solution of the differential equation

$$\prod_{r=1}^{m}f(\delta-r+1)y = x^m\prod_{r=1}^{m}g(\delta+r-1)y.$$

Since we may replace x by ωx without altering either x^m or the operator δ, it is evidently sufficient to prove the theorem for $y = y(x)$. Operations on this function give

$$\begin{aligned} f(\delta-1)f(\delta)y &= f(\delta-1)xg(\delta)y \\ &= xf(\delta)g(\delta)y \\ &= xg(\delta)xg(\delta)y = x^2g(\delta+1)g(\delta)y. \end{aligned}$$

Proceeding inductively in this way we obtain at length

$$f(\delta-m+1)\ldots f(\delta-1)f(\delta)y = x^m g(\delta+m-1)\ldots g(\delta+1)g(\delta)y,$$

and the proposition is proved.

If we are given a complete set of solutions of the equation $f(\delta) = xg(\delta)y$, the theorem will evidently, in general, furnish a complete set of solutions of the equation

$$\prod_{r=1}^{m} f(\delta-r+1)y = x^m \prod_{r=1}^{m} g(\delta+r-1)y.$$

2. The equation $f(\delta)y = x^m y$

We can apply theorem (4) with most satisfaction to differential equations of the form

$$(\delta-a_1)\ldots(\delta-a_p)y = x^m y, \tag{7}$$

and equations of this form furnish the examples which we discuss in the present note. We reserve for a second note the detailed discussion of equations of the more general form (1).

We confine ourselves to the case in which every root a_r of the indicial equation of (7) is real. Moreover, if any of the roots be negative, write $y = x^{-a}y'$ where a is positive and numerically greater than any negative root. We obtain an equation still of the form (7) but with every a_r positive.

Assuming, then, that every a_r is positive, we shall call the numbers a_r-ms, $(s = 1, 2,\ldots, I[a_r/m])$ the 'residues' of a_r, and the smallest of these, namely, the number $a_r-mI[a_r/m]$, we shall call the 'last residue of a_r'. If now we denote by b_r the 'last residue' of a_r and by b any residue of any of $a_1,\ldots,a_p$, then it follows from (4) that

$$y = \prod(\delta-b)z, \tag{8}$$

where the product is taken for every residue b of $a_1,\ldots,a_p$, is a solution of (7) whenever z is a solution of the differential equation

$$(\delta-b_1)(\delta-b_2)\ldots(\delta-b_p)z = x^m z. \tag{9}$$

Now the general solution of

$$\prod(\delta-b)z = 0 \tag{10}$$

is the finite form $z = \sum A_b x^b$,

where the A's are constants. But (9) has no solution of this type, for its form shows that any solution expressible as an ascending power-series may have a first term, $x^{b_r}(r = 1,\ldots,p)$, but can have no last term. The equations (9), (10) have therefore no common solution

and, if $z_1,\ldots,z_p$ be a complete set of solutions of (9), it is impossible to determine constants C_r such that

$$\prod(\delta-b)\sum_1^p C_r z_r = 0.$$

In consequence, the functions

$$\prod(\delta-b)z_r, \qquad (r=1,\ldots,p)$$

are a linearly independent set of solutions of (7). Thus the functions which enable us to solve (9) will, with their derivatives, furnish also the solution of (7), and we may therefore base our classification of equations (7) on the more restricted class (9), in which every b_r lies in the interval $(0, m)$.

Moreover, a transformation of the form $y = x^a Y$ allows us to impose an assigned value, say zero, on one of the b_r's so that it is the differences of the b_r's and not their absolute values which count in the classification.

As a first illustration of the foregoing theorems we consider the simplest class of these equations, namely, those which can be solved in terms of exponentials and polynomials: as a further illustration we discuss in § 4 equations soluble in terms of Bessel's Functions.

3. Equations soluble by exponentials

Suppose now that the order of the equation $f(\delta)y = x^m y$ is exactly m, and suppose further that the roots $a_1,\ldots,a_m$ of the indicial equation $f(a) = 0$ are all positive integers mutually incongruent (mod m) and, as a consequence, congruent (mod m) to the integers $0, 1,\ldots, m-1$.

Their 'last residues' $b_1,\ldots,b_m$ are then just the m integers $0, 1,\ldots, m-1$ in some order and equation (9) is now

$$\delta(\delta-1)\ldots(\delta-m+1)z = x^m z.$$

By elementary theory, or even as the simplest case of (6), this equation has the m solutions $z = e^{\omega x}$, where $\omega^m = 1$, and so the equation in y has the m solutions

$$y = \prod(\delta-b)e^{\omega x},$$

which are evidently of the form $ue^{\omega x}$ where u is some polynomial. We have therefore the theorem

(11) *If $a_1, a_2,\ldots,a_m$ are positive integers, all mutually incongruent (mod m), then the differential equation*

$$(\delta-a_1)\ldots(\delta-a_m)y = x^m y$$

can be solved in terms of exponentials and polynomials only, in the form

$$y = \prod(\delta - b)e^{\omega x}$$

where $\omega^m = 1$ *and the integers* b *are all the residues* $(\mathrm{mod}\, m)$ *of* $a_1, \ldots, a_m$.

With rather more generality we can evidently say that, if the differences $a_2 - a_1, \ldots, a_m - a_1$ are, in some order, congruent $(\mathrm{mod}\, m)$ to $1, 2, \ldots, m-1$, then the equation can be solved by expressions of the form $y = x^k u e^{\omega x}$ where u is a polynomial and k need be neither positive nor integral.

The simplest equation covered by (11) is the well-known equation

$$\delta(\delta - 2p - 1)y = x^2 y, \qquad (p \text{ a positive integer})$$

which, by (11), has the solutions

$$y = (\delta - 1)(\delta - 3)\ldots(\delta - 2p + 1)e^{\pm x}.$$

The relations which connect the operators δ and D enable us to deduce at once the four equivalent symbolic forms*

$$x^{2p+1}(x^{-1}D)^p x^{-1} e^{\pm x}, \qquad x^{2p+1}(x^{-1}D)^{p+1} e^{\pm x}$$

$$x^{-1}(x^3 D)^p x^{-(2p-1)} e^{\pm x}, \qquad x^{-3}(x^3 D)^{p+1} x^{-2p} e^{\pm x}.$$

Similarly, the third-order equation

$$\delta(\delta - 3p - 1)(\delta - 3q - 2)y = x^3 y, \qquad (p, q \text{ positive integers})$$

has the solutions

$$y = \prod_{r=0}^{p-1}(\delta - 3r - 1) \prod_{s=0}^{q-1}(\delta - 3s - 2)e^{\omega x}$$

where $\omega^3 = 1$. These give corresponding symbolic forms of which

$$x^{3q+2}(x^{-2}D)^q x^{3p-1}(x^{-2}D)^p x^{-1} e^{\omega x},$$

$$x^{-1}(x^4 D)^q x^{-(3q+1)}(x^4 D)^p x^{-(3p-2)} e^{\omega x},$$

are sufficiently typical.

We have been led, in another connexion, to consider differential equations of the type

$$\prod_{r=0}^{m-1}(\delta - rn)y = x^m y$$

where m, n are interprime positive integers. The integers rn $(r = 0, 1, \ldots, m-1)$ are then all mutually incongruent $(\mathrm{mod}\, m)$ and their residues b turn out to be the $\frac{1}{2}(m-1)(n-1)$ 'non-lattice'

* Watson, *Bessel Functions*, p. 110.

integers j, i.e. the positive integers which are not of the form $am+bn$ where a, b are positive integers or zero. The equation is therefore a case of (11) and its solution may be written as

$$y = \prod(\delta - j)e^{\omega x}$$

where $\omega^m = 1$.

As in (11) the solution is expressed as the product of $e^{\omega x}$ and a polynomial

$$u(x) = e^{-\omega x}\prod(\delta - j)e^{\omega x}.$$

There seems, in general, to be no convenient explicit expression for this polynomial. In the case of $n = 2$ there is, however, the known expansion*

$$u(x) = e^{-x}\prod_{r=0}^{m-1}(\delta - 2r - 1)e^{x}$$

$$= \sum_{r=0}^{m}\frac{(-)^r(m+r)!}{r!(m-r)!}\frac{x^{m-r}}{2^r}.$$

4. Equations soluble by Bessel's Functions

If, applying our methods to Bessel's equation of integral order n,

$$(\delta^2 - n^2)y = -x^2 y,$$

we make the substitution

$$y = x^{-n}\delta(\delta-2)\ldots(\delta-2n+2)z,$$

it is then sufficient that z be a solution of the equation

$$\delta^2 z = -x^2 z,$$

which is Bessel's equation of zero order.

Since Bessel's equation of zero or integral order n has the two solutions J_n, Y_n of which Y_n contains a logarithmic part, we must identify J_n and $x^{-n}\delta(\delta-2)\ldots(\delta-2n+2)J_0$.

By comparing leading coefficients we get actually

$$J_n = (-x)^{-n}\delta(\delta-2)\ldots(\delta-2n+2)J_0.$$

As before we may replace the δ-operator by symbolic operators in D and obtain the well-known formulae

$$J_n = (-x)^n(x^{-1}D)^n J_0 = (-x)^{-(n+2)}(x^2 D)^n x^{-(2n-2)}J_0.$$

The identification of Y_n and $x^{-n}\delta(\delta-2)\ldots(\delta-2n+2)Y_0$, which is less easy, leads to analogous results.

* Cf. Watson, loc. cit., p. 53.

More generally we can connect in this way Bessel's functions of orders μ, $\mu+m$ differing by an integer. For we have, by definition,

$$\{\delta^2-(\mu+m)^2\}J_{\mu+m} = -x^2J_{\mu+m}.$$

The substitution

$$J_{\mu+m} = (-x)^{-(\mu+m)}\prod_{r=0}^{m-1}(\delta-2\mu-2r)z$$

reduces the equation to the form

$$\delta(\delta-2\mu)z = -x^2z,$$

of which a solution is $z = x^\mu J_\mu(x)$. By comparison of leading coefficients we are led to the actual identification

$$J_{\mu+m}(x) = (-x)^{-m}\prod_{r=0}^{m-1}(\delta-\mu-2r)J_\mu(x)$$

which may be written in the equivalent forms,*

$$\begin{aligned} J_{\mu+m}(x) &= (-)^m x^{m+\mu}(x^{-1}D)^m x^{-\mu}J_\mu(x) \\ &= (-)^m x^{\mu-m-2}(x^3D)^m x^{-(\mu+2m-2)}J_\mu(x). \end{aligned}$$

For equations of higher order we deduce from (6), by writing x^2 for x and suppressing $g(\delta)$, that

(12) *The equation*

$$\prod_{r=0}^{m-1}f(\delta-2r)y = x^{2m}y$$

is soluble in terms of the functions which give the solution of the equation

$$f(\delta)y = x^2y.$$

Now the equation

$$(\delta-a)(\delta-b)y = x^2y$$

has the solutions

$$y = x^pJ_{\pm q}(ix), \qquad \text{where } p, q = \tfrac{1}{2}(a\pm b).$$

It follows that

(13) *The equation*

$$\prod_{r=0}^{m-1}(\delta-a-2r)(\delta-b-2r)y = x^{2m}y$$

is soluble by means of Bessel's functions of order $\frac{1}{2}(a-b)$.

More generally we deduce from (4) that

(14) *The equation*

$$\prod_{r=0}^{m-1}(\delta-a_r)(\delta-b_r)y = x^{2m}y$$

* Watson, loc. cit., p. 46.

is soluble by means of Bessel's functions of order $\frac{1}{2}(a-b)$, *if* a, b *are numbers such that, for* $r = 0, 1, \ldots, m-1$,

$$a_r - a \equiv 2r \equiv b_r - b \pmod{2m}.$$

In particular the equation

$$\prod_{r=1}^{m} (\delta - 2r - 2mN_r - a)(\delta - 2r - 2mN_r - b)y = x^{2m}y, \tag{15}$$

where $N_r (r = 1, \ldots, m)$ are integral or zero, is soluble by Bessel's functions of order $\frac{1}{2}(a-b)$.

Write $\left(\frac{x}{a-b}\right)^{a-b} = x'$ and drop accents. We obtain

$$\prod_{r=1}^{m} (\delta - p_r)(\delta - p_r - 1)y = x^p y, \tag{16}$$

where $(a-b)p_r = b + 2r + 2mN_r, \qquad (a-b)p = 2m.$

i.e. $$x^{p_1-p+2}D^2 x^{p_2-p_1+2}D^2 \ldots x^{p_m-p_{m-1}+2}D^2 x^{-p_m}y = y. \tag{17}$$

Since there are $m!$ possible rearrangements of the numbers p_r, there are $m!$ symbolic forms corresponding to (17), apparently distinct but in fact essentially the same. We collect these results for reference in the theorem

(18) *If, for* $r = 1, \ldots, m$, *the numbers* p_r *are, in some order, the numbers* $p[N_r + (\frac{1}{2}b + r)/m]$, *where the* N*'s are integral or zero and* b *is a number independent of* r, *then the equation*

$$\prod_{r=1}^{m} (\delta - p_r)(\delta - p_r - 1)y = x^p y,$$

or equivalently

$$x^{p_1-p+2}D^2 x^{p_2-p_1+2}D^2 \ldots x^{p_m-p_{m-1}+2}D^2 x^{-p_m}y = y,$$

can be solved by Bessel's functions of order m/p.

The equation of order 4 deserves rather closer study. We consider it in the form

$$\delta(\delta - a)(\delta - b)(\delta - a - b)y = x^4 y. \tag{19}$$

This is evidently soluble, if $b \equiv 2 \pmod 4$, in Bessel's functions of order $\frac{1}{2}a$ and so, by symmetry, in Bessel's functions of order $\frac{1}{2}b$, if $a \equiv 2 \pmod 4$. A particular case of this equation has been studied by Nicholson* and Watson.†

* *Proc. Royal Soc.* 93 A (1917), pp. 506–19.

† Watson, loc. cit., pp. 107–8.

Theirs is the equation

$$D^2x^{4\mu}D^2y = x^{2\mu}y, \tag{20}$$

i.e. $$\delta(\delta-1)(\delta+4\mu-2)(\delta+4\mu-3)y = x^{4-2\mu}y.$$

If we write $x^{1-\frac{1}{2}\mu} = (1-\frac{1}{2}\mu)x'$ and drop accents, the equation becomes

$$\delta\left(\delta-\frac{2}{2-\mu}\right)\left(\delta-\frac{4-8\mu}{2-\mu}\right)\left(\delta-\frac{6-8\mu}{2-\mu}\right)y = x^4y, \tag{21}$$

which we identify with (19) by writing

$$\frac{2}{2-\mu} = a, \qquad \frac{4-8\mu}{2-\mu} = b.$$

It is therefore soluble

(i) in Bessel's functions of order $\dfrac{2-4\mu}{2-\mu}$, if $\dfrac{2}{2-\mu} \equiv 2 \pmod 4$;

(ii) in Bessel's functions of order $\dfrac{1}{2-\mu}$, if $\dfrac{4-8\mu}{2-\mu} \equiv 2 \pmod 4$;

i.e. if (i) $\mu = \dfrac{4N+1}{2N+1}$, (ii) $\mu = \dfrac{4N}{2N-3}$

where N is an integer or zero. Watson* has given the solutions corresponding to $\mu = 1,\ 0,\ 3,\ \frac{4}{5}$, arising from $N = 0, -1$ in (i), (ii) above, but suggests that 'these seem to be the only equations of Nicholson's type which are soluble with the aid of Bessel functions'.

It may therefore be worth while to give the explicit (symbolical) solutions for any integral N. It is desirable to give separately the formulae for N positive or zero, and for N negative; and it is convenient to write $\mathfrak{I}_r$ for any solution of either of the Bessel's equations

$$(\delta^2-r^2)y = \pm x^2y.$$

The formulae are then

(i) $\mu = \dfrac{4N+1}{2N+1}$: (22)

$$y = x^{-(12N+2)}\prod_{r=1}^{N}(\delta-12N-4r)(\delta+12N-4r+4)\mathfrak{I}_{12N+2}$$

$$= x^{4N+2}(x^{-3}D)^N x^{-(20N+4)}(x^{-3}D)^N x^{12N}\mathfrak{I}_{12N+2};$$

(ii) $\mu = \dfrac{4N-1}{2N-1}$:

$$y = x^{8N}\prod_{r=1}^{N-1}(\delta+12N-4r)(\delta-12N-4r+4)\mathfrak{I}_{12N-2}$$

$$= (x^{-3}D)^{4N-1}x^{28N-8}(x^{-3}D)^{N-1}x^{-12N}\mathfrak{I}_{12N-2};$$

* Watson, loc. cit., p. 108.

(iii) $\mu = \dfrac{4N}{2N-3}$:

$$y = x^{-\frac{1}{2}N+\frac{1}{2}} \prod_{r=1}^{N} (\delta - \tfrac{1}{3}N - 4r + \tfrac{5}{2})(\delta + \tfrac{1}{3}N - 4r + \tfrac{3}{2}) \mathfrak{I}_{\frac{1}{2}N-\frac{1}{2}}$$
$$= x^{4N+2}(x^{-3}D)^N x^{\frac{1}{2}N+1}(x^{-3}D)^N x^{\frac{1}{2}N-\frac{1}{2}} \mathfrak{I}_{\frac{1}{2}N-\frac{1}{2}};$$

(iv) $\mu = \dfrac{4N}{2N+3}$:

$$y = x^{-\frac{1}{2}N+\frac{1}{2}} \prod_{r=1}^{N-1} (\delta + \tfrac{1}{3}N - 4r + \tfrac{5}{2})(\delta - \tfrac{1}{3}N - 4r + \tfrac{3}{2}) \mathfrak{I}_{\frac{1}{2}N+\frac{1}{2}}$$
$$= (x^{-3}D)^{N-1} x^{\frac{1}{2}N-3} (x^{-3}D)^{N-1} x^{-\frac{1}{2}N-\frac{1}{2}} \mathfrak{I}_{\frac{1}{2}N+\frac{1}{2}}.$$

In these formulae, N is always a positive integer in (ii), (iv), and a positive integer or zero in (i), (iii). Symbolic formulae may also be given involving powers of (x^5D) instead of powers of $(x^{-3}D)$.

If in (i), (iii) we write $N = 0$, and in (ii), (iv) $N = 1$, we obtain

(i) $N = 0$, $\quad \mu = 1$, $\quad y = x^{-2}\mathfrak{I}_2$; (23)

(ii) $N = 1$, $\quad \mu = 3$, $\quad y = x^8 \mathfrak{I}_{10}$;

(iii) $N = 0$, $\quad \mu = 0$, $\quad y = x^{\frac{1}{2}} \mathfrak{I}_{\frac{1}{2}}$;

(iv) $N = 1$, $\quad \mu = \frac{4}{5}$, $\quad y = x^{-\frac{1}{5}} \mathfrak{I}_{\frac{1}{5}}$,

which are equivalent to the results given by Watson.

SELF-RECIPROCAL FUNCTIONS

By G. H. HARDY *and* E. C. TITCHMARSH

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1. Introduction

1.1. THE Fourier reciprocity between two real even functions $f(x)$ and $g(x)$ is expressed by the formulae

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos xy f(y)\, dy, \qquad f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos xy\, g(y)\, dy, \quad (1.11)$$

and two functions so connected are said to be *cosine transforms* of one another; they were described by Cauchy as *reciprocal* functions.† The integrals may be integrals of the classical kind (as in Cauchy) or integrals in some generalized sense (as in the modern theory of transforms‡).

In special cases $f(x) = g(x)$, or

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos xy f(y)\, dy, \quad (1.12)$$

a singular homogeneous integral equation of the second kind. We then describe $f(x)$ as a *self-reciprocal* function. There are familiar examples such as

$$x^{-\frac{1}{2}}, \qquad e^{-\frac{1}{2}x^2}, \qquad \operatorname{sech} x\sqrt{(\tfrac{1}{2}\pi)};$$

a more novel one§ is

$$f(x) = x^{\frac{1}{2}} J_{-\frac{1}{4}}(\tfrac{1}{2}x^2).$$

1.2. Our problem is *to determine all self-reciprocal functions*, or rather (since complete generality is hardly attainable) all such functions of certain classes, such as the class L^2 of functions of integrable square.

It may be well to observe at once that there is a sense in which

† 'Of the first kind', reciprocal functions of the second kind being sine transforms of each other. See Cauchy (**8**), Burkhardt (**5**).

‡ For the standard results of this theory, which we shall take for granted in what follows, see Plancherel (**21, 22**), Titchmarsh (**25, 26**), Hobson (**16**), §§ 481–8.

§ See § **3** below.

there is an immediate solution. Suppose, to be definite, that $\phi(x)$ belongs to L^2, i.e. that $\phi(x)$ is measurable and

$$\int_0^\infty \phi^2(x)\,dx$$

finite. Then $\phi+\phi^*$, where the star denotes the transform, is also a function of L^2, and is self-reciprocal, since

$$(\phi+\phi^*)^* = \phi^*+\phi.$$

Also any self-reciprocal f may be expressed as $\frac{1}{2}f+\frac{1}{2}f = \frac{1}{2}f+(\frac{1}{2}f)^*$, so that the solution is complete.

On the other hand it is obvious that none of the examples we have quoted have been obtained in this way, and the solution does not enable us to decide (unless by actual verification) whether a given $f(x)$ is self-reciprocal. To determine whether $f(x)$ is of the form $\phi+\phi^*$ is in fact to solve another integral equation, viz.

$$f(x) = \phi(x)+\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos xy\,\phi(y)\,dy$$

(a non-homogeneous singular equation).

1.3. We therefore look for a solution of a different type; but it is convenient first to enlarge the formal basis of our analysis. The same problem arises for the equation

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin xy f(y)\,dy, \tag{1.31}$$

and, more generally, for

$$f(x) = \int_0^\infty \sqrt{(xy)}J_\nu(xy)f(y)\,dy, \tag{1.32}$$

where $J_\nu(x)$ is a Bessel function of order $\nu \geqslant -\frac{1}{2}$.† The cosine and sine cases are the cases in which $\nu = -\frac{1}{2}$, and $\nu = \frac{1}{2}$.

The functions

$$x^{-\frac{1}{2}}, \qquad xe^{-\frac{1}{2}x^2}, \qquad \frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}}$$

are self-reciprocal in the sine transform, and the functions

$$x^{-\frac{1}{2}}, \qquad x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}, \qquad x^{\frac{1}{2}}J_{\frac{1}{2}\nu}(\tfrac{1}{2}x^2)$$

in the transform of order ν.‡

† For the theory of 'Hankel transforms' of order $\nu \geqslant -\frac{1}{2}$ see Titchmarsh (25, 27), Plancherel (23). There does not seem to have been any discussion of the Hankel formulae for $\nu < -\frac{1}{2}$.

‡ The appropriate references are given in § 3.

1.4. We can obtain two formal solutions of our problem as follows. Here, and throughout the paper, we give details of the analysis for the equation (1.12) only. The general equation (1.32) provides us with a greater wealth of formal illustration, but presents no additional difficulty of principle, so that we are content with stating results.

(i) *First solution.* We multiply (1.12) by x^{s-1} and integrate from 0 to ∞ with respect to x. If we write

$$\phi(s) = \int_0^\infty x^{s-1} f(x)\, dx \tag{1.41}$$

and assume the convergence of all the integrals concerned, and the legitimacy of the inversion of the order of integration, we obtain

$$\phi(s) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty x^{s-1}\, dx \int_0^\infty \cos xy\, f(y)\, dy$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y)\, dy \int_0^\infty x^{s-1} \cos xy\, dx = \sqrt{\left(\frac{2}{\pi}\right)} \Gamma(s) \cos \tfrac{1}{2}s\pi \int_0^\infty y^{-s} f(y)\, dy,$$

or

$$\phi(s) = \sqrt{\left(\frac{2}{\pi}\right)} \Gamma(s) \cos \tfrac{1}{2}s\pi\, \phi(1-s). \tag{1.42}$$

If now we write

$$\phi(s) = 2^{\frac{1}{2}s} \Gamma(\tfrac{1}{2}s) \psi(s), \tag{1.43}$$

(1.42) becomes

$$\psi(s) = \psi(1-s). \tag{1.44}$$

Finally, inverting (1.41) by Mellin's formula,† we obtain

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) x^{-s}\, ds \tag{1.45}$$

or

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s} \Gamma(\tfrac{1}{2}s) \psi(s) x^{-s}\, ds, \tag{1.46; A}$$

where $\psi(s)$ is an even function of $s-\frac{1}{2}$. This is our first solution.

The simplest example is

$$\psi(s) = 1, \qquad f(x) = 2e^{-\frac{1}{4}x^2}.$$

Formal analysis on the general lines of that just set out is naturally to be found in many places in the literature,‡ though we have never

† For discussions of the Mellin formulae see Mellin (**17**), Hardy (**11**), Pollard (**24**), Burkill (**6**).

‡ See, for example, citations of Ramanujan by Hardy and Littlewood (**15**), § 2.53; or that of Schwarzschild by Eddington (**9**), 208–10, where everything is stated in terms of the Fourier inversion formulae, of which Mellin's formulae are, from the strictly formal point of view, merely a variant.

seen it applied to this particular problem, and the formula (*A*) appears to be new.

1.5. (ii) *Second solution.* Let

$$\lambda(s) = \int_0^\infty e^{-\frac{1}{2}sx^2} f(x)\, dx. \tag{1.51}$$

Then multiplying (1.12) by $e^{-\frac{1}{2}sx^2}$, integrating with respect to x, and again assuming the convergence of the integrals and the legitimacy of the inversion, we find

$$\lambda(s) = \int_0^\infty e^{-\frac{1}{2}sx^2} f(x)\, dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty e^{-\frac{1}{2}sx^2}\, dx \int_0^\infty \cos xy\, f(y)\, dy$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y)\, dy \int_0^\infty e^{-\frac{1}{2}sx^2} \cos xy\, dx = \frac{1}{\sqrt{s}} \int_0^\infty e^{-y^2/2s} f(y)\, dy$$

or
$$\lambda(s) = s^{-\frac{1}{2}} \lambda\left(\frac{1}{s}\right). \tag{1.52}$$

Hence, if
$$\mu(s) = s^{\frac{1}{2}} \lambda(s), \tag{1.53}$$

we have
$$\mu(s) = \mu\left(\frac{1}{s}\right). \tag{1.54}$$

Again
$$\lambda(s) = \int_0^\infty e^{-su} \frac{f\{\sqrt{(2u)}\}}{\sqrt{(2u)}}\, du = \int_0^\infty e^{-su} g(u)\, du, \tag{1.55}$$

say. There is again a well-known formula for the inversion of this equation, viz.†

$$g(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{us} \lambda(s)\, ds. \tag{1.56}$$

Substituting into (1.56) from (1.53), we obtain

$$f(x) = \frac{x}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s} s^{-\frac{1}{2}} \mu(s)\, ds, \tag{1.57; B}$$

where $\mu(s)$ is a solution of (1.54). This is our second solution. The simplest example is

$$\mu(s) = 1, \qquad f(x) = \frac{2^{\frac{1}{2}}}{\Gamma(\frac{1}{4})} \frac{1}{\sqrt{x}}.$$

† Apparently due, in this form, to Pincherle (**20**); see Bateman (**3**), 61. The formulae are, of course, again variants of the Fourier formulae.

The formulae corresponding to (A) and (B) for general ν are

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s} \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu + \tfrac{1}{4}) \psi(s) x^{-s}\, ds \qquad (1.58;\ A')$$

where $\psi(s)$ satisfies (1.44), and

$$f(x) = \frac{x^{\frac{1}{2}-\nu}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{4}x^2} s^{-\frac{1}{2}(\nu+1)} \mu(s)\, ds, \qquad (1.59;\ B')$$

where $\mu(s)$ satisfies (1.54).

1.6. In § 2 we discuss formula (A). Our main object is to find theorems which give a complete solution of the problem for stated classes of functions $f(x)$, that is to say theorems of the type 'in order that $f(x)$ should be a self-reciprocal function of a certain class P, it is necessary and sufficient that $f(x)$ should be given by the formula (A), $\psi(s)$ being a function of a certain class Q'. Theorems **1**, **3**, and **4** are of this type, the class P being in Theorem **1** the class L^2, in Theorem **3** an associated class of functions of a real variable, and in Theorem **4** a certain class (the most natural class for this problem) of analytic functions.

In § 3 we consider the known examples of self-reciprocal functions, and others of our own, in the light of the theorems we have proved. There are some of them which do not fall under any of the theorems, and we are thus led to Theorems **13** and **14**, in which the conditions are not of the 'necessary and sufficient' type, but which enable us to render an account of all our examples but one.

In § 6 we prove the theorem for our second solution (Theorem **16**) which corresponds to Theorem **4**. This theorem covers the example which evaded the first solution. Generally, the second solution is more powerful for analytic functions and the first for real functions. There is a theorem concerning the second solution analogous to Theorem **1**, but it does not seem to be so useful.

We also state, but without proof, a theorem (Theorem **17**) corresponding to Theorem **13**. Our general principle in selecting theorems has been to concentrate on 'necessary and sufficient' theorems, with the reservation that there should be at least one theorem to cover any well-known example.

2. The first solution

Functions of L^2

2.1. THEOREM 1. *A necessary and sufficient condition that a function $f(x)$ of $L^2(0, \infty)$† should be its own cosine transform is that it should be of the form* (1.45), *where $c = \frac{1}{2}$, the integral is a mean-square integral, and $\phi(\frac{1}{2}+it) = \chi(t)$ belongs to $L^2(-\infty, \infty)$ and satisfies* (1.42), *i.e.*

$$\frac{\chi(t)}{2^{\frac{1}{4}+\frac{1}{2}it}\Gamma(\frac{1}{4}+\frac{1}{2}it)} \tag{2.11}$$

is an even function of t.

(1) *The condition is necessary.* Since

$$\int_0^\infty f^2(x)\,dx = \int_{-\infty}^\infty f^2(e^v)e^v\,dy,$$

the function

$$F(y) = f(e^v)e^{\frac{1}{2}v}$$

belongs to $L^2(-\infty, \infty)$. It follows from the theory of transforms that

$$\chi(t) = \int_{-\infty}^\infty F(y)e^{ity}\,dy = \int_0^\infty f(x)x^{-\frac{1}{2}+it}\,dx$$

exists as a mean-square integral for almost all t. Also $\chi(t)$ belongs to $L^2(-\infty, \infty)$, and

$$F(y) \equiv \frac{1}{2\pi}\int_{-\infty}^\infty \chi(t)e^{-ity}\,dt,‡$$

the integral being again a mean-square integral. This formula is equivalent to (1.45).

It remains to verify that the function (2.11) is even. The formal proof of this is obtained by putting $s = \frac{1}{2}+it$ in the analysis of § 1.4. A direct justification of the inversion of the order of integration is troublesome, and we therefore proceed differently.

We have to prove that $\chi(t)$ is equivalent to

$$\omega(t) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(\tfrac{1}{2}+it)\cos(\tfrac{1}{4}\pi+\tfrac{1}{2}\pi it)\chi(-t),$$

and for this it is sufficient to prove that

$$\int_0^T \chi(t)\,dt = \int_0^T \omega(t)\,dt$$

† By $L^p(a, b)$ we mean the class of functions which belong to L^p in (a, b).

‡ $f(x) \equiv g(x)$ means that f is equivalent to g, i.e. that the functions differ at most in a set of measure zero.

for all T. Now $\chi(t)$ is the mean-square limit of

$$\int_{\xi}^{X} f(x)x^{-\frac{1}{2}+it}\,dx$$

when $\xi \to 0$, $X \to \infty$; and so

$$\int_0^T \chi(t)\,dt = \lim_{\xi\to 0,\, X\to\infty} \int_0^T dt \int_{\xi}^{X} f(x)x^{-\frac{1}{2}+it}\,dx$$

$$= \lim \int_{\xi}^{X} f(x)\,dx \int_0^T x^{-\frac{1}{2}+it}\,dt = \int_0^{\infty} f(x)\,dx \int_0^T x^{-\frac{1}{2}+it}\,dt.$$

Similarly

$$\int_0^T \omega(t)\,dt = \int_0^{\infty} f(x)\,dx \int_0^T \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(\tfrac{1}{2}+it)\cos(\tfrac{1}{4}\pi+\tfrac{1}{2}\pi it)x^{-\frac{1}{2}-it}\,dt;$$

and we have to prove that the last two repeated integrals are equal. Since f belongs to L^2, and is its own cosine transform, the result required is a case of Parseval's theorem for transforms, provided that the two functions

$$\int_0^T x^{-\frac{1}{2}+it}\,dt, \qquad \int_0^T \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(\tfrac{1}{2}+it)\cos(\tfrac{1}{4}\pi+\tfrac{1}{2}\pi it)x^{-\frac{1}{2}-it}\,dt$$

are cosine transforms of each other and belong to L^2.

The first function is

$$x^{-\frac{1}{2}}\frac{x^{iT}-1}{i\log x},$$

and therefore belongs to L^2. Its cosine transform is

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} \cos xu\,du \int_0^T u^{-\frac{1}{2}+it}\,dt,$$

while the second function is the result of inverting this integral. All that remains, then, is to justify this inversion; and since

$$\int_0^U \cos xu\,du \int_0^T u^{-\frac{1}{2}+it}\,dt = \int_0^T dt \int_0^U u^{-\frac{1}{2}+it}\cos xu\,du$$

for any finite U, the result will follow if only

$$\lim_{U\to\infty} \int_0^T dt \int_U^{\infty} u^{-\frac{1}{2}+it}\cos xu\,du = 0. \tag{2.12}$$

But for fixed x

$$\int_U^\infty u^{-\frac{1}{2}+it}\cos xu\,du = \left[u^{-\frac{1}{2}+it}\frac{\sin xu}{x}\right]_U^\infty + \frac{\frac{1}{2}-it}{x}\int_U^\infty u^{-\frac{1}{2}+it}\sin xu\,du$$

$$= O(U^{-\frac{1}{2}}) + O\left(\int_U^\infty u^{-\frac{3}{2}}\,du\right) = O(U^{-\frac{1}{2}})$$

uniformly for $0 \leqslant t \leqslant T$, and this proves (2.12).

(2) *The condition is sufficient.* In the first place, if $\chi(t)$ belongs to $L^2(-\infty, \infty)$, so does $F(y)$, and therefore $f(x)$ belongs to $L^2(0, \infty)$.

We have to prove that $g(x)$, the cosine transform of $f(x)$, is equivalent to $f(x)$. Now

$$g_1(x) = \int_0^x g(u)\,du = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\frac{\sin xy}{y}\,dy.$$

Since $(\sin xy)/y$ belongs to L^2, and $f(y)$ is the mean-square limit of

$$\frac{1}{2\pi}\int_{-T}^{T}\chi(t)y^{-\frac{1}{2}-it}\,dt,$$

it follows that

$$g_1(x) = \lim_{T\to\infty}\frac{1}{2\pi}\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{\sin xy}{y}\,dy\int_{-T}^{T}\chi(t)y^{-\frac{1}{2}-it}\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2\pi}\sqrt{\left(\frac{2}{\pi}\right)}\int_{-T}^{T}\chi(t)\,dt\int_0^\infty \frac{\sin xy}{y^{\frac{1}{2}+it}}\,dy,$$

since the last integral with respect to y is uniformly convergent in t. Hence

$$g_1(x) = \frac{1}{4}\sqrt{\left(\frac{2}{\pi}\right)}\int_{-\infty}^{\infty}\frac{\chi(t)}{\cos(\frac{1}{4}\pi+\frac{1}{2}\pi it)\Gamma(\frac{1}{2}+it)}\,\frac{x^{\frac{1}{2}+it}}{\frac{1}{2}+it}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(t)\frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt, \tag{2.15}$$

on using the functional equation for $\chi(t)$ and then changing the sign of t.

On the other hand

$$f_1(x) = \int_0^x f(u)\,du = \lim_{T\to\infty} \frac{1}{2\pi} \int_0^x du \int_{-T}^{T} \chi(t) u^{-\frac{1}{2}-it}\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \chi(t)\,dt \int_0^x u^{-\frac{1}{2}-it}\,du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(t)\, \frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt. \tag{2.16}$$

Comparing (2.15) and (2.16), we see that $g_1(x) = f_1(x)$, and so $g(x) \equiv f(x)$. This completes the proof of Theorem **1**.

Functions of L^p $(1 < p < 2)$

2.2. We suppose now that $1 < p < 2$, and denote by p' the index conjugate to p, that is to say the index defined by

$$p' = \frac{p}{p-1}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

THEOREM **2**. *If a function $f(x)$ of $L^p(0, \infty)$, where $1 < p < 2$, is its own cosine transform, then $f(x)$ is of the form* (1.45); *where $\phi(s)$ is an analytic function which* (i) *is regular in the strip*

$$1/p' < \sigma < 1/p, \tag{2.21}$$

(ii) *tends to zero uniformly when s tends to infinity inside any interior strip, and* (iii) *satisfies the equation* (1.42); *and the integral in* (1.45) *is a mean-square integral along any line of the strip* (2.21).

It will be observed that Theorem **2** is a one-sided theorem with conditions which are necessary only and not sufficient.

If $f(x)$ belongs to L^p, its cosine transform belongs to $L^{p'}$, so that $f(x)$ here belongs both to L^p and to $L^{p'}$, and therefore to all intermediate classes L^q. In particular it belongs to L^2, and so satisfies the conditions of Theorem **1**.

(i) The function $\phi(s)$ defined by (1.41) reduces to the $\chi(t)$ of Theorem **1** when $s = \frac{1}{2} + it$, but $\phi(s)$ is now analytic and regular in the strip (2.21). In fact

$$\int_0^\infty |f(x)| x^{\sigma-1}\,dx \leqslant \Big(\int_0^1 |f|^{p'}\,dx\Big)^{1/p'} \Big(\int_0^1 x^{p(\sigma-1)}\,dx\Big)^{1/p} +$$

$$+ \Big(\int_1^\infty |f|^p\,dx\Big)^{1/p} \Big(\int_1^\infty x^{p'(\sigma-1)}\,dx\Big)^{1/p'},$$

and these integrals converge for the values of σ stated. It follows in

the usual manner that $\phi(s)$ is regular in the strip and bounded in any interior strip.

(ii) We can write $\phi(s)$ in the form

$$\phi(s) = \Big(\int_0^\delta + \int_\delta^\Delta + \int_\Delta^\infty\Big) f(x)x^{\sigma-1}x^{it}\,dx = \phi_1(s)+\phi_2(s)+\phi_3(s).$$

Suppose now that $\eta > 0$ and

$$\frac{1}{p'}+\eta \leqslant \sigma \leqslant \frac{1}{p}-\eta. \tag{2.23}$$

Then

$$|\phi_1(s)| \leqslant \Big(\int_0^\delta |f|^{p'}\,dx\Big)^{1/p'}\Big(\int_0^\delta x^{p(\sigma-1)}\,dx\Big)^{1/p} = O(\delta^\eta)$$

when $\delta \to 0$, and we can therefore choose δ so that $|\phi_1| < \epsilon$ for all s of (2.21). Similarly, we can choose Δ so that $|\phi_3| < \epsilon$. When δ and Δ are fixed, $\phi_2 \to 0$ uniformly when $s \to \infty$ in (2.21). It follows that $\phi \to 0$ uniformly in (2.21).

(iii) It follows from Theorem **1** that $\phi(s)$ satisfies (1.42) on the line $s = \frac{1}{2}+it$, and therefore throughout (2.21).

Thus $\phi(s)$ possesses the properties stated in the theorem, and it remains only to prove (1.45). By Theorem **1**, (1.45) is true for $c = \frac{1}{2}$, so that it is sufficient to prove that the value of the integral is independent of c.

If c and c' are two values of σ in (2.21), and

$$f(x, c, T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \phi(s)x^{-s}\,ds,$$

then, by Cauchy's theorem,

$$f(x, c, T) - f(x, c', T) = \frac{1}{2\pi i}\int_{c-iT}^{c'-iT} \phi(s)x^{-s}\,ds - \frac{1}{2\pi i}\int_{c+iT}^{c'+iT} \phi(s)x^{-s}\,ds.$$

Since $\phi(s) \to 0$, this difference tends to zero when $T \to \infty$, and uniformly in any interval (a, b) of positive values of x. Hence

$$\lim_{T\to 0}\int_a^b \{f(x, c, T) - f(x, c', T)\}^2\,dx = 0,$$

so that $f(x, c, T)$ and $f(x, c', T)$ converge in mean to the same limit $f(x)$ independent of c.

Functions of L_p^* $(1<p<2)$

2.3. Theorem **2** is a one-sided theorem, and it is perhaps unreasonable, in view of the asymmetry of the theory of transforms about the number 2, to expect in this case a theorem as satisfactory as Theorem **1**. There is, however, a very similar class of functions for which we can obtain a complete solution.

We shall say that $f(x)$ belongs to $L_p^*(0,\infty)$, where $1<p<2$, if $x^\alpha f(x)$ belongs to $L_2(0,\infty)$ for

$$-\alpha_0=-\frac{1}{p}+\frac{1}{2}<\alpha<\frac{1}{p}-\frac{1}{2}=\alpha_0. \tag{2.31}$$

It is plain that $f(x)$ then belongs to $L^q(0,1)$ for $q\leqslant 2$. Suppose now that $p<q<2$. Then we can choose $\alpha<\alpha_0$ so that $2q\alpha>2-q$; and then

$$\int_1^\infty |f|^q\,dx\leqslant\Big(\int_1^\infty x^{2\alpha}|f|^2\,dx\Big)^{q/2}\Big(\int_1^\infty x^{-2q\alpha/(2-q)}\,dx\Big)^{(2-q)/2}<\infty,$$

so that $f(x)$ belongs to L^q. If also $f(x)$ is its own cosine transform, it belongs to $L^{q'}$, so that a self-reciprocal $f(x)$ of L_p^* belongs to all Lebesgue classes between L^p and $L^{p'}$ (though not usually to either of these).

The class of self-reciprocal functions of L_p^* is thus in this respect a little wider than the class of those of L^p. In other respects it is narrower. Suppose, for example, that $h(x)$ is defined by

$$h(x)=2^{-n}\ (n!-1\leqslant x\leqslant n!+1,\ n=2,3,\ldots)$$

and $h(x)=0$ elsewhere. Then $h(x)$ belongs to L^r for every positive r, but to no L_p^*, since $\sum 2^{-rn}$ is convergent but $\sum(n!)^{2\alpha}2^{-2n}$ divergent for every positive α. The cosine transform of $h(x)$ is

$$h^*(x)=\sqrt{\Big(\frac{2}{\pi}\Big)}\sum 2^{-n}\int_{n!-1}^{n!+1}\cos xy\,dy=2\sqrt{\Big(\frac{2}{\pi}\Big)}\frac{\sin x}{x}\sum 2^{-n}\cos n!x,$$

which is continuous and $O(x^{-1})$ at infinity, so that $h^*(x)$ belongs to L^r for $r>1$ and to L_p^* for $1<p<2$. Thus $h(x)+h^*(x)$ is a self-reciprocal function which belongs to L^r for all $r>1$, but to no L_p^*.

THEOREM 3. *A necessary and sufficient condition that a function $f(x)$ of $L_p^*(0,\infty)$ should be its own cosine transform is that it should be of the form* (1.45), *where $\phi(s)$ satisfies the conditions* (i), (ii), (iii) *of Theorem* **2** *and* (iv) *belongs to $L^2(-\infty,\infty)$, qua function of t, for all σ of* (2.21).

(1) *The condition is necessary.* Since $f(x)$ belongs to L^r for $p<r<p'$, we have only to show that $\phi(s)$ satisfies condition (iv). This results immediately from the theory of transforms, since

$$\phi(s)=\int_0^\infty x^{-\frac{1}{2}+it}.x^{\sigma-\frac{1}{2}}f(x)\,dx,$$

and $x^{\sigma-\frac{1}{2}}f(x)$ belongs to L^2 if $|\sigma-\frac{1}{2}|<\alpha_0$, i.e. if $1/p'<\sigma<1/p$.

(2) *The condition is sufficient.* Since $\phi(s)$ belongs to L^2 on the line $s=c+it$, the integral (1.45) exists as a mean-square integral for all c in question, and (as in § 2.2) its value is independent of c. It therefore defines a function $f(x)$ independent of c. Since

$$x^{c-\frac{1}{2}}f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(c+it)x^{-\frac{1}{2}-it}\,dt$$

for $|c-\frac{1}{2}|<\alpha_0$, and the right-hand side belongs to L^2 for every such c, $f(x)$ belongs to L_p^*. Finally, by Theorem **1**, $f(x)$ is self-reciprocal.

Analytic functions

2.4. We shall say that $f(x)$ belongs to $A(\omega, a)$, where $0<\omega\leqslant\pi$,† $a<\frac{1}{2}$, if (i) it is an analytic function of $x=re^{i\theta}$ regular in the angle A defined by $r>0$, $|\theta|<\omega$, and (ii) it is $O(|x|^{-a-\delta})$ for small x, and $O(|x|^{a-1+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta|\leqslant\omega-\eta<\omega$.

THEOREM **4**. *A necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be its own cosine transform is that it should be of the form (A); where $\psi(s)$ is regular, and satisfies* (1.44), *in the strip*

$$a<\sigma<1-a; \tag{2.41}$$

$$\psi(s)=O\{e^{(\frac{1}{2}\pi-\omega+\eta)|t|}\} \tag{2.42}$$

for every positive η and uniformly in any strip interior to (2.41); *and c is any value of σ in* (2.41).

(1) *The condition is necessary.* The integral (1.41) is absolutely convergent for $a<\sigma<1-a$, so that $\phi(s)$ is regular in (2.41). Also $f(x)$ belongs to L^2, and $\phi(\frac{1}{2}+it)$ is the $\chi(t)$ of Theorem **1**. It follows from Theorem **1** that $\phi(s)$ satisfies (1.42) on $\sigma=\frac{1}{2}$ and therefore throughout (2.41), or (what is the same thing) that $\psi(s)$ satisfies (1.44).

We may deform the integral (1.41), by Cauchy's theorem, into one

† This restriction on ω is not essential, but when $\omega>\pi$ the angle is re-entrant, and $f(x)$ is not usually one-valued in it.

along any radius vector $\theta = \beta$, where $|\beta| < \omega$, and we may suppose that β has the sign of t. Then

$$|\phi(s)| = \left| e^{i\beta\sigma - \beta t} \int_0^\infty r^{\sigma+it-1} f(re^{i\beta})\, dr \right| \leqslant e^{-\beta t} \int_0^\infty r^{\sigma-1} |f(re^{i\beta})|\, dr = O(e^{-|\beta|\,|t|}).$$

Since $|\beta|$ is any number less than ω, and

$$|\Gamma(\tfrac{1}{2}s)| \sim Ce^{-\frac{1}{4}\pi|t|} |\tfrac{1}{2}t|^{\frac{1}{2}\sigma-\frac{1}{2}}, \tag{2.43}$$

where C is a constant, this implies (2.42).

The integral (A) is absolutely convergent, and uniformly convergent for any fixed x and all c of any interval interior to (2.41). It represents $f(x)$ when $c = \frac{1}{2}$, by Theorem **1**, and therefore generally.

(2) *The condition is sufficient.* The modulus of the integrand in (A) is, by (2.42) and (2.43), less than a constant multiple of

$$r^{-\sigma} e^{-|t|(\omega-\eta-|\theta|)} |t|^{\frac{1}{2}c-\frac{1}{2}}.$$

The integral is therefore uniformly convergent throughout any domain of x interior to the angle A; and $f(x)$ is regular in A, and $O(|x|^{-c})$ in any smaller angle. Since c may be taken as near to a or $1-a$ as we please, $f(x)$ belongs to $A(\omega, a)$. Finally, $f(x)$ belongs to L^2, and is therefore, by Theorem **1**, self-reciprocal.

2.5. All of Theorems **2**–**4** have been deduced from Theorem **1** (though Theorem **4** might naturally be proved independently), and their proofs depend upon the theory of transforms, of which we shall make no further use. This is therefore a convenient place to consider some of the most interesting special cases in the light of what we have proved. In order to have a greater variety of illustrations, we state first the theorems corresponding to Theorems **1**–**4** for functions reciprocal in the Hankel transform.

THEOREMS **5**–**8**. *Theorems* **1**–**4** *are true for Hankel transforms with the following formal changes. In Theorem* **1** *the function* (2.11) *is to be replaced by*

$$\frac{\chi(t)}{2^{\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}it}\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}it)}. \tag{2.51}$$

In Theorems **2**, **3**, *and* **4**, (1.43) *is to be replaced by*

$$\phi(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\psi(s) \tag{2.52}$$

and (A) *by* (A').†

† Since $\nu \geqslant -\frac{1}{2}$, $\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})$ has no poles for which $\sigma > 0$, so that conditions concerning the regularity of $\phi(s)$ and $\psi(s)$ in the strips considered are equivalent.

There is a corresponding series of theorems for *skew-reciprocal* functions, i.e. functions for which $f(x) = -g(x)$, where $g(x)$ is the cosine or Hankel transform of $f(x)$.

THEOREMS **9–12**. *The results of all the preceding theorems become true for skew-reciprocal functions if we change the sign of the right-hand side of the functional equations satisfied by* $\chi(t)$, $\phi(s)$, *and* $\psi(s)$.

3. Examples

(1) If we take $\psi(s) = 1$ in Theorems **4** and **8**, and evaluate the integrals (A) and (A') by the calculus of residues, we obtain

$$f(x) = 2e^{-\frac{1}{2}x^2}, \qquad f(x) = 2^{\frac{3}{2}-\frac{1}{2}\nu}x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}$$

respectively. Conversely, we may assume these forms of $f(x)$, and calculate $\phi(s)$ and $\psi(s)$ by (1.41) and (1.43) or (2.52). The conditions of Theorems **4** or **8** (and *a fortiori* those of the less special theorems) are satisfied in these cases.

(2) If we take $\psi(s) = P(\frac{1}{2}-s)$, where $P(u)$ is an even polynomial, or an even integral function of order less than 1, we find that

$$f(x) = 2\sum_0^\infty \frac{(-\frac{1}{2}x^2)^n}{n!} P(2n+\tfrac{1}{2})$$

is its own cosine transform. If $P(u)$ is a polynomial, $f(x) = e^{-\frac{1}{2}x^2}Q(x^2)$, where $Q(u)$ is a polynomial. The simplest example, after $P(u) = 1$, is $P(u) = u^2$, when

$$f(x) = 2e^{-\frac{1}{2}x^2}(x^4-3x^2+\tfrac{1}{4}).$$

If $P(u)$ is odd we obtain examples of Theorem **12**, such as $e^{-\frac{1}{2}x^2}(x^2-\frac{1}{2})$.

(3) If $f(x) = \operatorname{sech} x\sqrt{(\frac{1}{2}\pi)}$, we find that

$$\phi(s) = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}s}\Gamma(s)L(s) = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}s}\Gamma(s)(1^{-s}-3^{-s}+5^{-s}-\ldots),$$

which satisfies (1.42). This is an example of Theorem **4**.

If
$$f(x) = \frac{1}{e^{x\sqrt{(2\pi)}}-1} - \frac{1}{x\sqrt{(2\pi)}},$$

we find that $\phi(s) = (2\pi)^{-\frac{1}{2}s}\Gamma(s)\zeta(s)$. Taking $\nu = \frac{1}{2}$, we obtain

$$\psi(s) = \frac{\xi(s)}{s(s-1)},$$

where $\xi(s)$ is Riemann's ξ-function. This is an example of Theorem **8**, with $\nu = \frac{1}{2}$.

(4) If† $$f(x) = x^{\frac{1}{2}-\nu} e^{\frac{1}{2}ix^2} \int_x^\infty t^{2\nu-1} e^{-\frac{1}{2}it^2}\, dt,$$

$f(x)$ is $O(x^{\frac{1}{2}-|\nu|})$ for small x‡ and $O(x^{\nu-\frac{3}{2}})$ for large x. The function satisfies the conditions of Theorem **8** when $\nu < 1$, but not when $\nu \geqslant 1$. In this case we find

$$\psi(s) = 2^{\frac{1}{2}\nu-\frac{7}{4}} \frac{\Gamma(\frac{1}{4}-\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(1-\nu)}.$$

This function satisfies the conditions of Theorem **8** when $\nu < 1$, but a pole comes on to the line $\sigma = \frac{1}{2}$ for $\nu = 1$. The integral (A') is convergent if $\nu < 1$ and $\nu - \frac{1}{2} < c < \frac{3}{2} - \nu$, but is never convergent if $\nu > 1$.

In the case $\nu = -\frac{1}{2}$ the function is

$$f(x) = 1 - xe^{\frac{1}{2}ix^2} \int_x^\infty e^{-\frac{1}{2}it^2}\, dt = \int_0^\infty e^{-\frac{1}{2}iw^2 - xw}\, w\, dw.$$

(5) If§ $$f(x) = \int_0^\infty \frac{e^{-xw}}{\cosh bw}\, dw,$$

where $b = \sqrt{(\frac{1}{2}\pi)}$, and $\nu = \frac{1}{2}$, we find that

$$\psi(s) = \frac{2^{\frac{3}{2}-s}\pi^{\frac{1}{2}+\frac{1}{2}s} L(1-s)}{\sin s\pi\, \Gamma(\frac{1}{2}+s)},$$

where $L(s)$ is defined as under (3). This is an example of Theorem **8**, with $\nu = \frac{1}{2}$. Here

$$f(x) = 2\left(\frac{1}{x+b} - \frac{1}{x+3b} + \frac{1}{x+5b} - \dots\right)$$

is regular except for $x = -b, -3b, \dots$, and is $O(1)$ for small and $O(1/x)$ for large x. Hence we may take $\omega = \pi$ and $a = 0$ in Theorem **8**; and $\psi(s)$ is regular in $0 < \sigma < 1$, with poles at $s = 0$ and $s = 1$, and is $O\{e^{-(\frac{1}{2}\pi-\eta)|t|}\}$, for every positive η, for large t.

(6) Suppose that‖

$$f(x) = x^{\frac{1}{2}+\nu} \int_0^\infty w^\nu F(w) e^{-\frac{1}{2}ix^2w^2}\, dw,$$

where $F(w) = F(1/w)$, and that

$$F(w) = O(w^\lambda) \quad (w \to 0), \qquad F(w) = O(w^{-\lambda}) \quad (w \to \infty)$$

† For the case $\nu = \frac{1}{2}$ see Hardy (**10**), Phillips (**19**). The general formula was given by Bailey (**1**).

‡ Except when $\nu = 0$, when there is an additional factor $\log x$.

§ Phillips (**19**).

‖ For this example see Bailey (**2**).

where $0<\lambda<1+\nu$.† It may easily be verified that $f(x)=f(re^{i\theta})$ is regular inside the angle $|\theta|\leqslant\frac{1}{4}\pi$, and that

$$f(x)=O(r^{-\frac{1}{2}+\lambda}) \quad (r\to 0), \qquad f(x)=O(r^{-\frac{1}{2}-\lambda}) \quad (r\to\infty)$$

uniformly in any smaller angle. Thus the conditions of Theorem **8** are satisfied with $\omega=\frac{1}{4}\pi$, $a=\frac{1}{2}-\lambda$.

We find by straightforward calculation that

$$\psi(s)=2^{\frac{1}{2}\nu-\frac{1}{2}}\int_0^\infty w^{-\frac{1}{2}-s}F(w)\,dw$$

if $\frac{1}{2}-\lambda<\sigma<\frac{1}{2}+\lambda$. It easily results that $\psi(s)$ is regular in this strip, bounded in any interior strip, and satisfies (1.44).

(7) A very interesting example of a more recondite character is

$$f(x)=x^{\frac{1}{2}-\nu}(x^2-b^2)^{\frac{1}{2}(\nu-1)}J_{\frac{1}{2}(\nu-1)}\{b\sqrt{(x^2-b^2)}\} \quad (x>b>0)$$
$$f(x)=0 \qquad (0<x<b).$$

That this function is its own Hankel transform results, by elementary substitutions and specializations, from a formula of Sonine.‡

Here $f(x)$ is $O\{(x-b)^{\frac{1}{2}\nu-\frac{1}{2}}\}$ near $x=b$, and $O(x^{-\frac{1}{2}\nu-\frac{1}{2}})$ at infinity, and it belongs to L^2 if $\nu>0$, in which case only do our theorems apply. The conditions of Theorem **3** are satisfied if $\nu>2\alpha_0$.

To calculate $\psi(s)$ we observe that

$$\phi(s)=\int_b^\infty x^{s-\frac{1}{2}-\nu}(x^2-b^2)^{\frac{1}{2}(\nu-1)}J_{\frac{1}{2}(\nu-1)}\{b\sqrt{(x^2-b^2)}\}\,dx$$
$$=b^{\frac{1}{2}s+\frac{1}{4}\nu}\int_0^\infty(1+u^2)^{\frac{1}{2}(s-\nu-\frac{3}{2})}u^{\frac{1}{2}(\nu+1)}J_{\frac{1}{2}(\nu-1)}(b^2u)\,du.$$

This integral may be calculated by another formula of Sonine,§ and we find that

$$\psi(s)=b^{\nu-\frac{1}{2}}2^{\frac{1}{4}-\frac{1}{4}\nu}\frac{K_{\frac{1}{2}(s-\frac{1}{2})}(b^2)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}.$$

From this, and the fact that $K_\mu(x)$ is an even function of μ, it follows that $\psi(s)$ satisfies (1.44).

(8) Taking $\qquad f(x)=x^{\frac{1}{2}}J_{\frac{1}{4}\nu}(\frac{1}{2}x^2),$

we find $\qquad \psi(s)=\dfrac{\pi^{\frac{1}{2}}2^{-\frac{1}{4}\nu-\frac{1}{2}}}{\Gamma(\frac{1}{4}\nu+\frac{5}{8}+\frac{1}{4}s)\Gamma(\frac{1}{4}\nu+\frac{7}{8}-\frac{1}{4}s)},$

† The analysis can be adapted by slight changes to the case $\lambda>1+\nu$, but our theorems fail if $\lambda<0$.

‡ Watson (**30**), § 13.47 (1).

§ See Watson (**30**), § 13.6 (2).

so that $\psi(s) = \psi(1-s)$. Since

$$\int_0^\infty tJ_\nu(xt)J_{\frac{1}{4}\nu}(\tfrac{1}{2}t^2)\,dt = J_{\frac{1}{4}\nu}(\tfrac{1}{2}x^2),\dagger$$

$f(x)$ is its own Hankel transform. In this case, however, $f(x)$ behaves like $x^{\nu+\frac{1}{2}}$ for small x and like $x^{-\frac{1}{2}}$ for large x, and does not belong to L^2, or to any L^p for which $p \leqslant 2$, so that it is not an example of any of the theorems so far proved.

The integral (A') is convergent if $-\nu-\frac{1}{2} < \sigma < \frac{5}{2}$ (absolutely if $-\nu-\frac{1}{2} < \sigma < \frac{1}{2}$), and $\phi(s)$ satisfies the conditions of our theorems concerning regularity, but its order of magnitude for large t is that of $|t|^{\frac{1}{2}\sigma-\frac{3}{4}}$, and it does not belong to L^2 on $\sigma = \frac{1}{2}$.

(9) There are very interesting examples of self-reciprocal functions in the analytic theory of numbers. Suppose that $r(n)$ is the number of representations of n as a sum of two squares, and that

$$\overline{\mathrm{P}}(x) = \sum_{0<n<x}{}' r(n) - \pi x, \qquad f(x) = x^{-\frac{3}{4}}\left\{\overline{\mathrm{P}}\left(\frac{x^2}{2\pi}\right)-1\right\}, \tag{3.1}$$

the dash implying the insertion of a factor $\frac{1}{2}$ in the last term of the sum when x is an integer. Then $f(x)$ *is its own Hankel transform when* $\nu = 2$. In fact the well-known identity

$$\overline{\mathrm{P}}(x) = \sqrt{x}\sum_1^\infty \frac{r(n)}{\sqrt{n}} J_1\{2\pi\sqrt{(nx)}\} \tag{3.2}$$

is easily transformed by partial summation into the form

$$\int_0^\infty \frac{1}{y}\left\{\overline{\mathrm{P}}\left(\frac{y^2}{2\pi}\right)-1\right\}J_2(xy)\,dy = \frac{1}{x^2}\left\{\overline{\mathrm{P}}\left(\frac{x^2}{2\pi}\right)-1\right\}, \tag{3.3}$$

in which it expresses the property stated.‡

Here $f(x) = O(x^{\frac{1}{2}})$ for small x. If we use the known (but rather difficult) theorem that $\overline{\mathrm{P}}(x) = O(x^{\frac{1}{4}})$ for large x, then $f(x) = O(x^{-\frac{3}{4}})$ for large x, and $f(x)$ satisfies the conditions of Theorem **7** for $\nu = 2$ and $\alpha_0 = \frac{1}{3}$.

Calculation shows that

$$\phi(s) = \tfrac{1}{2}(2\pi)^{\frac{1}{2}s-\frac{3}{4}}\frac{Z(\frac{3}{4}-\frac{1}{2}s)}{\frac{3}{4}-\frac{1}{2}s},$$

† This is a trivial transformation of a formula due to Bateman. See Bateman (**4**), 185, Hardy (**13**), and Watson (**30**), 437.

‡ See Hardy (**14**). It should have been mentioned in this paper that the formula had been found independently by A. L. Dixon, who communicated it to Hardy when the paper was completed but not in print.

where $Z(s)$ is the analytic function defined for $\sigma > 1$ by

$$Z(s) = \sum \frac{r(n)}{n^s} = \zeta(s)(1^{-s} - 3^{-s} + 5^{-s} - \ldots) = \zeta(s)L(s),$$

and that $\psi(s)$ satisfies (1.44).

It should be observed that we can hardly hope to *prove* the difficult identity (3.2) as a corollary of a general theorem of this character. What results from Theorem **7** is the existence of the integral (3.3) as a mean-square integral; what is required to prove (3.2) is its convergence in the ordinary sense.

It has been proved by Walfisz and Oppenheim that, if $r_p(n)$ is the number of representations of n as a sum of p squares, and $\overline{\mathrm{P}}_p(x)$ is defined as in (3.1), then

$$\overline{\mathrm{P}}_p(x) = x^{\frac{1}{2}p} \sum_1^\infty \frac{r_p(n)}{n^{\frac{1}{2}p}} J_{\frac{1}{2}p}\{2\pi\sqrt{(nx)}\},$$

the series being summable by Cesàro's means of sufficiently high order.† The corresponding result in our present language would be that

$$x^{-\frac{1}{2}-\frac{1}{4}p}\left\{\overline{\mathrm{P}}_p\left(\frac{x^2}{2\pi}\right) - 1\right\}$$

is its own Hankel transform of order $1+\frac{1}{2}p$. If we take $p = 3$, and use Walfisz's result $\overline{\mathrm{P}}_3(x) = O(x^{\frac{3}{4}+\epsilon})$,‡ we find that $f(x)$ falls under Theorem **5**. This is not true for any larger p.

If we take $p = 1$, we find that

$$f(x) = \frac{1}{x}\left\{\frac{x}{\sqrt{(2\pi)}} - \left[\frac{x}{\sqrt{(2\pi)}}\right]\right\}$$

is its own Hankel transform of order $\frac{3}{2}$, as may be verified directly.

(10) The functions

$$f(x) = x^{-\frac{1}{2}}, \qquad f(x) = 2^{\frac{1}{2}a}\Gamma(\tfrac{1}{2}a)x^{-a} + 2^{\frac{1}{2}(1-a)}\Gamma(\tfrac{1}{2} - \tfrac{1}{2}a)x^{a-1}$$

are their own cosine transforms. Neither of them belongs to L^2 (or to any other Lebesgue class). The integral (1.41) is not convergent for any s. It is summable by an appropriate adaptation of Riesz's logarithmic means, i.e. as

$$\lim_{\lambda\to\infty} \frac{1}{\log\lambda} \int_1^\lambda \frac{dw}{w} \int_{1/w}^w f(x)x^{s-1}\,dx,$$§

† For a precise statement of the final results see Oppenheim **(18)**, Walfisz **(28)**.

‡ Walfisz **(29)**.

§ Compare Pollard **(24)**, 463.

but its value is 0 except for certain special values of s† and then is ∞. It is clear that our theorems fail to give any account of these solutions.

4. Further theorems concerning the first solution

4.1. There are several of the preceding examples which are not covered by any of the theorems which we have proved. Our next theorem, which was suggested by a theorem of Pollard,‡ is a one-sided theorem, which asserts only that reciprocal functions of a certain class have necessarily a certain form; but the class in question is a good deal more general than those which we have considered so far.

THEOREM 13. *Suppose that*

(i) *the integrals*

$$\int_0^1 |f(x)|\,dx, \qquad \int_1^\infty \frac{|f(x)|}{x^2}\,dx \tag{4.11}$$

are finite;

(ii) *the integral* (1.41), *defined as a Cauchy integral at both limits, is convergent for* $|\sigma-\frac{1}{2}|<\alpha\leqslant\frac{1}{2}$;

(iii) *the integral* (1.12), *defined as a Cauchy integral at infinity, is convergent, and equal to* $f(x)$, *for all positive* x (*so that* $f(x)$ *is finite for all positive* x *and is its own cosine transform*).

Then $f(x)$ *is of the form* (A), *where the integral is summable* $(C, 1)$ *for* $|c-\frac{1}{2}|<\alpha$, *and where* $\psi(s)$ *is regular, and satisfies* (1.44), *for* $|\sigma-\frac{1}{2}|<\alpha$.

The integral (1.41) is uniformly convergent in any strip interior to $|\sigma-\frac{1}{2}|<\alpha$, so that it represents an analytic function $\phi(s)$ regular for $|\sigma-\frac{1}{2}|<\alpha$. We have next to prove that $\phi(s)$ satisfies (1.42).

In what follows we suppose that s has a fixed value in the strip $|\sigma-\frac{1}{2}|<\alpha$. Let

$$f_1(x)=\int_0^x f(u)\,du, \qquad f_2(x)=\int_0^x f_1(u)\,du.$$

Then, by partial integration,

$$\int_\delta^X x^{s-1}f(x)\,dx = X^{s-1}f_1(X)-(s-1)X^{s-2}f_2(X)-$$

$$-\delta^{s-1}f_1(\delta)+(s-1)\delta^{s-2}f_2(\delta)+(s-1)(s-2)\int_\delta^X x^{s-3}f_2(x)\,dx. \tag{4.12}$$

† $s=\frac{1}{2}$ in the first case, $s=a$ and $s=1-a$ in the second.

‡ Pollard (**24**).

Let
$$g(x) = \int_0^x u^{s-1} f(u)\, du.$$

Then
$$f_1(\delta) = \int_0^\delta u^{1-s} \cdot u^{s-1} f(u)\, du = \delta^{1-s} g(\delta) - (1-s) \int_0^\delta u^{-s} g(u)\, du$$
$$= o(\delta^{1-\sigma}) + \int_0^\delta o(u^{-\sigma})\, du = o(\delta^{1-\sigma})$$

when $\delta \to 0$; and therefore also $f_2(\delta) = o(\delta^{2-\sigma})$. Hence the integrated terms in (4.2) which involve δ tend to zero with δ. Similarly, using the function
$$h(x) = \int_x^\infty u^{s-1} f(u)\, du$$

we can show that the integrated terms which involve X tend to zero when $X \to \infty$. It follows that, if $\phi(s)$ is defined by (1.41),

$$\phi(s) = (s-1)(s-2) \int_0^\infty x^{s-3} f_2(x)\, dx. \tag{4.13}$$

4.2. Let
$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y) \frac{1-\cos xy}{y^2}\, dy. \tag{4.21}$$

The integral is absolutely and uniformly convergent, in virtue of condition (i), so that $F(x)$ is continuous. Also, if $x > 0$ and h is sufficiently small,

$$\frac{F(x+2h) + F(x-2h) - 2F(x)}{4h^2} = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y) \cos xy \left(\frac{\sin hy}{hy}\right)^2 dy;$$

and the limit of this, when $h \to 0$, is the value of the integral (1.12),† i.e. $f(x)$. It follows that
$$f_2(x) = F(x). \tag{4.22}$$

Substituting from (4.21) and (4.22) in (4.13), we obtain

$$\phi(s) = (s-1)(s-2) \sqrt{\left(\frac{2}{\pi}\right)} \lim \int_\delta^X x^{s-3}\, dx \int_0^\infty f(y) \frac{1-\cos xy}{y^2}\, dy$$
$$= (s-1)(s-2) \sqrt{\left(\frac{2}{\pi}\right)} \lim \int_0^\infty \frac{f(y)}{y^2}\, dy \int_\delta^X x^{s-3} (1-\cos xy)\, dx, \tag{4.23}$$

† The argument which proves this is familiar, and the details are given by Pollard (**24**), 457–8.

the inversion being justified by the uniform convergence of the integral with respect to y. We wish to take the limit under the integral sign, and to justify this we must show that

$$J_1 = \int_0^\infty \int_X^\infty \to 0, \qquad J_2 = \int_0^\infty \int_0^\delta \to 0 \tag{4.24}$$

when $X \to \infty$ and $\delta \to 0$ respectively.

(i) We write

$$J_1 = \int_0^1 \int_X^\infty + \int_1^\infty \int_X^\infty = J_3 + J_4.$$

Then $J_4 \to 0$ because the integral with respect to x converges uniformly for $y \geqslant 1$. Also

$$J_3 = \int_0^1 \frac{f(y)}{y^2}\, dy \int_X^\infty x^{s-3}(1-\cos xy)\, dx = \int_0^1 \frac{f(y)}{y^s}\, dy \int_{Xy}^\infty u^{s-3}(1-\cos u)\, du.$$

If now we write

$$G(y) = \int_0^y z^{-s} f(z)\, dz$$

and integrate by parts, we obtain

$$J_3 = G(1) \int_X^\infty u^{s-3}(1-\cos u)\, du + \int_0^1 G(y)(Xy)^{s-3}(1-\cos Xy)\, X dy.$$

The first term plainly tends to zero. To prove that the second does so also, suppose that $A = A(s)$ is the maximum of $|G(y)|$, and that $|G(y)| < \epsilon$ for $y \leqslant \eta$. Then the modulus of the second term does not exceed

$$\epsilon \int_0^\eta (Xy)^{\sigma-3}(1-\cos Xy)\, X dy + A \int_\eta^1 (Xy)^{\sigma-3}(1-\cos Xy)\, X dy$$

$$= \epsilon \int_0^{\eta X} u^{\sigma-3}(1-\cos u)\, du + A \int_{\eta X}^\infty u^{\sigma-3}(1-\cos u)\, du,$$

which may be made as small as we please by choice of ϵ, η, and X in succession.

(ii) We write

$$J_2 = \int_0^1 \int_0^\delta + \int_1^\infty \int_0^\delta = J_5 + J_6,$$

and $J_5 \to 0$ because $1-\cos xy \leqslant \frac{1}{2}x^2y^2$. If we write

$$H(y) = \int_y^\infty z^{-s} f(z)\, dz,$$

we have

$$J_6 = \int_1^\infty \frac{f(y)}{y^2}\, dy \int_0^\delta x^{s-3}(1-\cos xy)\, dx = \int_1^\infty \frac{f(y)}{y^s}\, dy \int_0^{\delta y} u^{s-3}(1-\cos u)\, du$$

$$= H(1)\int_0^\delta u^{s-3}(1-\cos u)\, du + \int_1^\infty H(y)(\delta y)^{s-3}(1-\cos \delta y)\, \delta dy,$$

and from this point the proof proceeds substantially as before. We may therefore take the limit under the integral sign in (4.23).

We have thus proved that

$$\phi(s) = (s-1)(s-2)\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{f(y)}{y^2}\, dy \int_0^\infty x^{s-3}(1-\cos xy)\, dx,$$

and, inserting the value of the inner integral, we see that $\phi(s)$ satisfies (1.42).

Finally (A) requires no further proof, since it follows from a form of Mellin's inversion theorem already proved by Hardy.†

4.3. We may generalize Theorem **13** as follows :

THEOREM **14**. *The conclusions of Theorem* **13** *are still true if condition* (iii) *is satisfied except for a finite number of values of* x, *provided that* $f(x) \to 0$ *when* $x \to \infty$.

The only part of the proof which requires reconsideration is the proof that $F(x) = f_2(x)$. This is still true when $F(x)$ ceases to have a generalized second derivative for a finite number of values of x, provided that $F(x+2h)+F(x-2h)-2F(x) = o(h)$ for the exceptional values of x, that is to say here if

$$\int_0^\infty f(y)\cos xy \frac{\sin^2 hy}{y^2}\, dy = o(h);$$

and the integral is

$$O\left\{h^2\int_0^{1/h} |f(y)|\, dy + \int_{1/h}^\infty \frac{|f(y)|}{y^2}\, dy\right\} = o(h),$$

since $f(y) \to 0$.

4.4. We may test Theorems **13** and **14** on those of the examples

† Hardy (**11**), Theorem A.

of § 3 which escaped our earlier theorems and in which $\nu = -\frac{1}{2}$. These are (7), (8), and (10).

Taking (8) first, so that $f(x) = x^{\frac{1}{2}}J_{-\frac{1}{4}}(\frac{1}{2}x^2)$, we have $f(x) = O(1)$ for small x and $f(x) = O(x^{-\frac{1}{2}})$ for large x, so that (i) is satisfied. The integral (1.41) is convergent in this case if $0 < \sigma < \frac{5}{2}$, so that (ii) is satisfied with $\alpha = \frac{1}{2}$. Finally, the integral in (iii) is convergent for all positive x. This example therefore falls under Theorem **13**.

In (7), $$f(x) = x(x^2-b^2)^{-\frac{3}{4}}J_{-\frac{3}{4}}\{b\sqrt{(x^2-b^2)}\}$$ for $x > b$, and behaves at infinity like a multiple of $x^{-\frac{1}{2}}\cos(bx+\frac{1}{8}\pi)$. Condition (i) is satisfied, and condition (ii) is satisfied if $0 < \sigma < \frac{5}{4}$, so that we may take $\alpha = \frac{1}{2}$. Condition (iii) is satisfied except for $x = b$. Thus Theorem **13** fails, but the example falls under Theorem **14**.

Finally, the examples under (10) still evade our theorems, since the integral (1.41) is not convergent for any s. We shall return to these examples later, considering first some extensions in another direction.

5. Stieltjes integrals

5.1. Extensions of the Fourier formula to Stieltjes integrals have been given by various writers, notably by Hahn, N. Wiener, and Burkill. The extension with which we are concerned now is Burkill's.† Burkill proves that *if $f_1(x)$ is of bounded variation in any finite interval $(0, X)$; if*

$$f_1(0) = 0, \qquad f_1(x) = \tfrac{1}{2}\{f_1(x+0)+f_1(x-0)\} \quad (x > 0); \qquad (5.11)$$

if

$$\int^{\infty} \frac{|df_1(x)|}{x}$$

is finite; and if

$$g_1(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} \frac{\sin xt}{t}\, df_1(t);$$

then

$$f_1(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty} \frac{\sin xt}{t}\, dg_1(t).$$

The theorem may be illustrated by the example

$$f_1(x) = 0, \tfrac{1}{2}\lambda, \lambda, \tfrac{1}{2}\lambda, 0 \quad (x < a, x = a, a < x < b, x = b, x > b);$$

here

$$g_1(x) = \lambda\left(\frac{\sin ax}{a} - \frac{\sin bx}{b}\right).$$

It suggests that we should consider the circumstances in which $f_1(x)$ and $g_1(x)$ are identical.

† Burkill (**6**).

5.2. THEOREM **15.** *Suppose that $f_1(x)$ has bounded variation in any finite interval $(0, X)$, and satisfies* (5.11).

Suppose also that (i) *the integral*

$$\int_1^\infty \frac{|df_1(x)|}{x^2} \tag{5.21}$$

is finite; (ii) *the integral*

$$\phi(s) = \int_0^\infty x^{s-1}\, df_1(x) = \lim_{\delta\to 0,\, X\to\infty} \int_\delta^X x^{s-1}\, df_1(x) \tag{5.22}$$

is convergent for $|\sigma-\frac{1}{2}| < \alpha \leqslant \frac{1}{2}$; (iii) *the integral*

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin xy}{y}\, df_1(y) = \sqrt{\left(\frac{2}{\pi}\right)} \lim_{Y\to\infty} \int_0^Y \frac{\sin xy}{y}\, df_1(y)$$

is convergent, and equal to $f_1(x)$, for all positive x.

Then

$$f_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s)\, \frac{x^{1-s}}{1-s}\, ds = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \phi(s)\, \frac{x^{1-s}}{1-s}\, ds,$$

where $|c-\frac{1}{2}| < \alpha$ *and $\phi(s)$ has the properties stated in Theorem* **13**.

It is plain that $\phi(s)$ is still regular in $|\sigma-\frac{1}{2}| < \alpha$. The partial integrations of § 4.1 are still valid, and (4.13) is still true.

Now let
$$G(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{xy - \sin xy}{y^3}\, df_1(y). \tag{5.23}$$

Then $G(x)$ is continuous, and

$$\frac{G(x+2h)+G(x-2h)-2G(x)}{4h^2} = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin xy}{y} \left(\frac{\sin hy}{hy}\right)^2 df_1(y). \tag{5.24}$$

If we put

$$H(y) = \int_y^\infty \frac{\sin xu}{u}\, df_1(u),$$

and integrate by parts, we see that the integral in (5.24) is equal to

$$\sqrt{\left(\frac{2}{\pi}\right)} H(0) + \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty H(y)\, \frac{d}{dy} \left(\frac{\sin hy}{hy}\right)^2 dy$$

$$= f_1(x) + \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty H\left(\frac{u}{h}\right) \frac{d}{du} \left(\frac{\sin u}{u}\right)^2 du. \tag{5.25}$$

Now $H(y)$ is bounded, and tends to 0 when $y \to \infty$, and

$$\int_0^\infty \left| \frac{d}{du}\left(\frac{\sin u}{u}\right)^2 \right| du$$

is convergent. It follows that the integral in (5.25) tends to zero with h, and so, from (5.24), that $G(x)$ has a generalized second derivative equal to $f_1(x)$ for all positive x. Since $G(x)$ has also a continuous first derivative equal to

$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{1-\cos xy}{y^2}\, df_1(y),$$

it follows that $F(x) = f_2(x)$, as in (4.22).

It now follows from (4.13) that

$$\frac{\phi(s)}{(s-1)(s-2)} = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty x^{s-3}\, dx \int_0^\infty \frac{1-\cos xy}{y^2}\, df_1(y),$$

and we can deduce, substantially as in § 4, that $\phi(s)$ satisfies (1.42).

5.3. We cannot in this case complete the proof by an appeal to a known theorem. We have

$$\int_{c-iT}^{c+iT} \phi(s)\frac{x^{1-s}}{1-s}\, ds = \int_{c-iT}^{c+iT} \frac{x^{1-s}}{1-s}\, ds \int_0^\infty u^{s-1} df_1(u) = \int_0^\infty df_1(u) \int_{c-iT}^{c+iT} \left(\frac{x}{u}\right)^{1-s} \frac{ds}{1-s}. \tag{5.31}$$

the inversion being justified by the uniform convergence of the integral with respect to u. We divide this into three parts corresponding to the intervals $(0, \delta)$, (δ, Δ), and (Δ, ∞) of u; we suppose $\delta < x < \Delta$. The inner integral on the right-hand side of (5.31) is bounded for $\delta \leqslant u \leqslant \Delta$ and all T,† and its limit when $T \to \infty$ is $2\pi i$ for $u < x$ and 0 for $u > x$. Hence

$$\lim_{T\to\infty} \int_\delta^\Delta df_1(u) \int_{c-iT}^{c+iT} \left(\frac{x}{u}\right)^{1-s} \frac{ds}{1-s} = 2\pi i \int_\delta^x df_1(u).$$

Our proof will be completed if we can show that the corresponding integrals over $(0, \delta)$ and (Δ, ∞) tend to zero when $\delta \to 0$ and $\Delta \to \infty$, uniformly in T.

† This is a familiar result, and may be proved directly by applying Cauchy's theorem to a contour formed by the path of integration and a semicircle described on it as diameter.

Since $\frac{1}{2}-\alpha<c$, we can choose β so that $\frac{1}{2}-\alpha<\beta<c$. Let

$$\chi(u)=\int_0^u v^{\beta-1}\,df_1(v).$$

Then

$$\int_0^\delta df_1(u)\int_{c-iT}^{c+iT}\left(\frac{x}{u}\right)^{1-s}\frac{ds}{1-s}=\int_0^\delta u^{\beta-1}f_1(u)\,du\int_{c-iT}^{c+iT}x^{1-s}u^{s-\beta}\frac{ds}{1-s} \tag{5.32}$$

$$=\chi(\delta)\int_{c-iT}^{c+iT}x^{1-s}\delta^{s-\beta}\frac{ds}{1-s}-\int_0^\delta\chi(u)\,du\int_{c-iT}^{c+iT}x^{1-s}u^{s-\beta-1}\frac{s-\beta}{1-s}\,ds.$$

We can estimate the integrals with respect to s by applying Cauchy's theorem to the line $(c-iT,\ c+iT)$ and the perpendiculars through its ends in the direction of positive infinity. We find thus that

$$\int_{c-iT}^{c+iT}x^{1-s}\delta^{s-\beta}\frac{ds}{1-s}=2\pi i\delta^{1-\beta}+O\Big(\delta^{c-\beta}\int_c^\infty x^{-\sigma}\,d\sigma\Big)=O(\delta^{c-\beta}),$$

uniformly in T; and also that

$$\int_{c-iT}^{c+iT}x^{1-s}u^{s-\beta-1}\frac{s-\beta}{1-s}\,ds=2\pi i(1-\beta)u^{-\beta}+O\Big(\int_c^\infty x^{1-\sigma}u^{\sigma-\beta-1}\,d\sigma\Big)$$

$$=O(u^{-\beta})+O\left\{\frac{u^{c-\beta-1}}{\log(x/u)}\right\}=O(u^{c-\beta-1})$$

when $u\to 0$, uniformly in T. Hence (5.32) is

$$\chi(\delta)O(\delta^{c-\beta})+\int_0^\delta\chi(u)O(u^{c-\beta-1})\,du=o(\delta^{c-\beta})$$

and tends to zero with δ, uniformly in T.

The integral involving Δ may be treated similarly; and this completes the proof of the theorem.

5.4. To obtain an example of Theorem **15**, suppose that $a=\sqrt{(2\pi)}$ and that

$$f_1(x)=\frac{x}{a}-\left[\frac{x}{a}\right],\qquad f_1(x)=\tfrac{1}{2},$$

according as x is not or is a multiple of a. It may be verified at once that conditions (i) and (ii) of Theorem **15** are satisfied. Also

$$\int_0^\infty\frac{\sin xy}{y}\,df_1(y)=\frac{1}{a}\int_0^\infty\frac{\sin xy}{y}\,dy-\sum_1^\infty\frac{\sin nax}{na}=\sqrt{(\tfrac{1}{2}\pi)}f_1(x),$$

so that $g_1(x)=f_1(x)$.

In this case

$$\phi(s) = \lim_{N\to\infty} \int_0^{(N+\frac{1}{2})a} x^{s-1}\, df_1(x) = \lim_{N\to\infty} \left\{ \frac{1}{a} \int_0^{(N+\frac{1}{2})a} x^{s-1}\, dx - \sum_1^N (na)^{s-1} \right\}$$

$$= \lim_{N\to\infty} a^{s-1} \left\{ \frac{(N+\frac{1}{2})^s}{s} - \sum_1^N n^{s-1} \right\} = -(2\pi)^{\frac{1}{2}s-\frac{1}{2}} \zeta(1-s),$$

and so satisfies (1.42).

5.5. This example may be expressed more picturesquely in a slightly different manner. Let us agree to write $f(x) = f_1'(x)$ in any interval in which $f_1(x)$ is an integral, and

$$f(x) = \lambda\infty$$

at a discontinuity of $f_1(x)$ where the saltus is λ. Further, let us agree that $\int \phi f\, dx$ means $\int \phi\, df_1$, and that

$$\int_0^\infty \phi(y)\cos xy\, dy = \mu\infty$$

means that

$$\int_0^{\pi Y} \phi(y)\cos xy\, dy \sim \mu Y$$

when $Y\to\infty$.

In our example

$$f(x) = -1\infty \ (x = na), \qquad f(x) = 1/a \ (x \neq na),$$

where $a = \sqrt{(2\pi)}$. An easy calculation shows that

$$\int_0^{\pi Y} f(y)\cos xy\, dy = \frac{\sin \pi Y x}{ax} + \frac{1}{2} - \frac{\sin(\nu+\frac{1}{2})ax}{2\sin\frac{1}{2}ax},$$

where ν is the greatest integer such that $\nu a \leqslant \pi Y$, and the last two terms are to be replaced by $-\nu$ if x is a multiple of a. It follows that

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y)\cos xy\, dy = \frac{1}{a} \quad (C,1)$$

if $x \neq na$, and

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y)\cos xy\, dy = -\sqrt{\left(\frac{2}{\pi}\right)} \sqrt{\left(\frac{\pi}{2}\right)} \infty = -1\infty$$

if $x = na$. In other words, $f(x)$ is its own cosine transform.

All this is in the main merely a translation into other language of what Theorem **15** asserts in this particular case. There is, however,

nothing in Theorem **15** to justify any general assertion concerning the summability $(C, 1)$ of

$$\int_0^\infty \cos xy \, df_1(y).$$

5.6. The examples in § 3 (10), which do not fall under any of the theorems proved so far, may also be fitted into our scheme by appropriate conventions. Take, for example,

$$f(x) = x^{-\frac{1}{2}}, \qquad \phi(\tfrac{1}{2}+it) = \int_0^\infty x^{-\frac{1}{2}-it}\, dx.$$

This integral is never convergent or summable by a mean of Cesàro type, but it is summable by 'logarithmic means'†, except when $t = 0$, and has the value 0. It amounts to the same thing to say that, if we effect the formal transformation $x = e^u$, we obtain

$$\phi(\tfrac{1}{2}+it) = \int_{-\infty}^\infty e^{-itu}\, du,$$

and that the last integral is summable $(C, 1)$ to 0 except when $t = 0$, when it is infinite.

The integral is the formal derivative of the integral

$$\Phi(t) = i \int_{-\infty}^\infty \frac{e^{-itu}}{u}\, du = \pi \operatorname{sgn} t$$

and we may interpret the formula (1.45) as meaning

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty x^{-\frac{1}{2}-it}\, d\Phi(t) = \frac{1}{\sqrt{x}}.$$

The function $\phi(\frac{1}{2}+it)$, which is zero except for $t = 0$, and then infinite, is in a sense a solution of (1.42).

Similar interpretations may be found for the other solutions mentioned in § 3 (10). These functions are, however, interpreted much more naturally as examples of our second general solution, and we shall not attempt to fit our formalities into exact theorems.

6. The second solution

6.1. We shall say that $f(x)$ belongs to $A^*(\omega, a)$, where $0 < \omega \leqslant \frac{1}{4}\pi$, $0 < a < \frac{1}{2}$, if (i) it is an analytic function of $x = re^{i\theta}$ regular in the angle A^* defined by $r > 0$, $|\theta| < \omega$, and (ii) it is $O(|x|^{-a-\frac{1}{2}-\delta})$ for small

† For a formal definition see the end of § 3.

x, and $O(|x|^{a-\frac{1}{2}+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta| \leqslant \omega-\eta < \omega$.

It is to be observed that a function of $A^*(\omega, a)$, unlike the functions of § 2.4, does not usually belong to L^2, so that we cannot base our analysis on the theory of transforms.

6.2. THEOREM **16**. *A necessary and sufficient condition that a function $f(x)$ of $A^*(\omega, a)$ should be its own cosine transform is that it should be of the form (B), where c is any positive number, the integral is the limit of an integral over $(c-iT, c+iT)$, and $\mu(s)$ has the properties*

(i) *$\mu(s) = \mu(\rho e^{i\phi})$ is an analytic function of s, regular in the angle $B(\omega, a)$ defined by $\rho > 0$, $|\phi| < \frac{1}{2}\pi+2\omega$;*

(ii) *$\mu(s)$ is $O(|s|^{-\frac{1}{2}a-\delta})$ for small s, and $O(|s|^{\frac{1}{2}a+\delta})$ for large s, for every positive δ and uniformly in any angle $|\phi| \leqslant \frac{1}{2}\pi+2\omega-\zeta < \frac{1}{2}\pi+2\omega$;*

(iii) *$\mu(s)$ satisfies the equation* (1.54) *in $B(\omega, a)$.*

(i) *The condition is necessary.* The integrals

$$\int_0^1 |f|\,dx, \qquad \int_1^\infty \frac{|f|}{x}\,dx$$

are finite, and (*ex hypothesi*) the integral (1.12) converges for every positive x to the integrable value $f(x)$. The conditions of Pollard's theorem quoted in § 4.2 are therefore satisfied, and (in the notation of § 4.2) $F(x) = f_2(x)$. We may differentiate this equation with respect to x, since the resulting integral is uniformly convergent. Hence

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\frac{\sin xy}{y}\,dy = f_1(x). \tag{6.21}$$

We define $\lambda(s)$ and $\mu(s)$ as in § 1.5. It is plain first that these functions are regular for $\sigma > 0$. We may move the path of integration to the line $\theta = \theta_1$ of A^*†, provided that $\cos(\phi+2\theta) > 0$ for all θ between 0 and θ_1. Then

$$\lambda(s) = e^{i\theta_1}\int_0^\infty \exp(-\tfrac{1}{2}sr^2e^{2i\theta_1})f(re^{i\theta_1})\,dr,$$

and this formula shows that $\lambda(s)$ is regular in any angle in which $\cos(\phi+2\theta_1) > 0$. Varying θ_1 in A^*, we see that $\lambda(s)$ and $\mu(s)$ are regular in B.

† We use A^* both for the angle and for the class of functions associated with it.

Suppose now that s is in the angle of condition (ii). Then $\lambda(s)$ possesses a majorant of the type

$$\int_0^\infty e^{-C\rho r^2}|f(re^{i\theta})|\,dr,$$

where $C = C(\theta_1, \zeta)$. For small ρ this is

$$O\Big(\int_0^\infty e^{-C\rho r^2} r^{a-\frac{1}{2}+\delta}\,dr\Big) = O(\rho^{-\frac{1}{2}a-\frac{1}{4}-\frac{1}{2}\delta}),$$

and for large ρ it is

$$O\Big(\int_0^\infty e^{-C\rho r^2} r^{-a-\frac{1}{2}-\delta}\,dr\Big) = O(\rho^{\frac{1}{2}a-\frac{1}{4}+\frac{1}{2}\delta}).\dagger$$

Hence $\mu(s)$ satisfies condition (ii). It remains to verify (1.54) and (B).

We suppose now that $\sigma = c > 0$. Integrating by parts, and using (6.21), we find

$$\lambda(s) = s\int_0^\infty e^{-\frac{1}{2}sx^2} x f_1(x)\,dx = \sqrt{\Big(\frac{2}{\pi}\Big)}\, s\int_0^\infty e^{-\frac{1}{2}sx^2} x\,dx \int_1^\infty f(y)\frac{\sin xy}{y}\,dy.$$

The double integral is absolutely convergent, by comparison with

$$\int_0^\infty e^{-\frac{1}{2}cx^2} x^2\,dx \int_0^1 |f|\,dy + \int_0^\infty e^{-\frac{1}{2}cx^2} x\,dx \int_1^\infty \frac{|f|}{y}\,dy,$$

and we may invert the integrations. We thus obtain

$$\lambda(s) = \sqrt{\Big(\frac{2}{\pi}\Big)}\, s\int_0^\infty \frac{f(y)}{y}\,dy \int_0^\infty e^{-\frac{1}{2}sx^2} x \sin xy\,dx$$

$$= s^{-\frac{1}{2}}\int_0^\infty f(y) e^{-y^2/2s}\,dy = s^{-\frac{1}{2}}\lambda\Big(\frac{1}{s}\Big),$$

which is equivalent to (1.54).

Finally, putting $x = \sqrt{(2y)}$ and $f(x) = x g(y)$, we obtain

$$\lambda(s) = \int_0^\infty e^{-sy}\frac{f(x)}{x}\,dy = \int_0^\infty e^{-su} g(y)\,dy,$$

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{ys}\lambda(s)\,ds = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{ys}\,ds \int_0^\infty e^{-su} g(u)\,du.$$

† Large r are important for small ρ, and conversely.

The inner integral is uniformly convergent, and so, inverting the order of integration, we obtain

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{ys}\lambda(s)\,ds = \frac{1}{\pi}\int_0^\infty e^{c(y-u)}\frac{\sin T(y-u)}{y-u}g(u)\,du.$$

Since $g(u) = O(u^{-\frac{1}{2}a-\frac{1}{2}-\frac{1}{2}\delta})$ for small u, the integral

$$\int_0^\infty e^{-cu}|g(u)|\,du$$

is finite. Also $g(u)$ is analytic for $u = y$. It therefore follows from the classical theory of Dirichlet's integral that

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} e^{ys}\lambda(s)\,ds = g(y);$$

and this is equivalent to (B).

(ii) *The condition is sufficient.* Suppose $f(x)$ defined by (B). We begin by deforming the integral into one taken along the radii $|\phi| = \frac{1}{2}\pi+\lambda$, where $2|\theta| < \lambda < 2\omega$. The integral then becomes absolutely convergent, and has a majorant of the type

$$O\Big(r\int_0^1 e^{-C\rho r^2}\rho^{-\frac{1}{2}a-\frac{1}{2}-\delta}\,d\rho\Big)+O\Big(r\int_1^\infty e^{-C\rho r^2}\rho^{\frac{1}{2}a-\frac{1}{2}+\delta}\,d\rho\Big).$$

It follows that $f(x)$ is regular in $A^*(\omega, a)$, and is

$$O\{r\,.\,(r^2)^{-\frac{1}{2}a-\frac{1}{2}-\delta}\} = O(r^{-a-\frac{1}{2}-2\delta})$$

for small r, and $O(r^{a-\frac{1}{2}-2\delta})$ for large r. Hence $f(x)$ belongs to $A^*(\omega, a)$.

6.3. It is to be observed that the results just proved concerning the order of $f(x)$, and those assumed concerning that of $\mu(s)$, carry with them as corollaries corresponding results concerning the derivatives of these functions. In fact

$$f^{(n)}(x) = O(|x|^{-a-\frac{1}{2}-n-2\delta}), \qquad f^{(n)}(x) = O(|x|^{a-\frac{1}{2}-n+2\delta})$$

for small and large x respectively, uniformly in any angle interior to A^* (and in particular for real x); and similarly

$$\mu^{(n)}(s) = O(|s|^{-\frac{1}{2}a-n-\delta}), \qquad \mu^{(n)}(s) = O(|s|^{\frac{1}{2}a-n+\delta})$$

for small and large s, uniformly in any angle $|\phi| \leqslant \frac{1}{2}\pi+2\omega-\zeta$ (and in particular on the line $\sigma = c$).

Suppose, for example, that x is positive and small, and let C be a circle in the u-plane with its centre at $u = x$ and touching the lines $|\mathrm{am}\, u| = \omega - \delta$. Then

$$f^{(n)}(x) = \frac{n!}{2\pi i}\int_C \frac{f(u)}{(u-x)^{n+1}}\,du = O\left(\frac{x^{-a-\frac{1}{4}-2\delta}.x}{x^{n+1}}\right) = O(x^{-a-\frac{1}{4}-n-2\delta}).$$

Similarly in the other cases.

Taking in particular $n = 1$, and integrating by parts, we see that the integral (1.12) is uniformly convergent in any finite interval of positive values of x; and similarly the integral (B) is uniformly convergent in any similar interval.

6.4. We now define $\lambda(s)$ by (1.51), and prove that it satisfies (1.53). Suppose that $\Re(u) = v > 0$, take the integral (B) along the contour Γ used at the end of § 6.2, and substitute into (1.51). We thus obtain

$$\lambda(u) = \frac{1}{2\pi i}\int_0^\infty xe^{-\frac{1}{2}ux^2}\,dx \int_\Gamma e^{\frac{1}{2}x^2 s} s^{-\frac{1}{4}}\chi(s)\,ds.$$

Here the integral is absolutely convergent. This is obvious so far as concerns the finite part of Γ, while on the distant parts there is a convergent majorant of the type

$$\int_0^\infty xe^{-\frac{1}{2}vx^2}\,dx \int_{\mathrm{P}}^\infty e^{-\frac{1}{2}Cx^2\rho}\rho^{\frac{1}{2}a-\frac{1}{4}+\delta}\,d\rho = \int_{\mathrm{P}}^\infty \frac{\rho^{\frac{1}{2}a-\frac{1}{4}+\delta}\,d\rho}{v+C\rho}.$$

We may therefore invert the integrations, and so obtain

$$\lambda(u) = \frac{1}{2\pi i}\int_\Gamma \frac{s^{-\frac{1}{4}}\mu(s)}{u-s}\,ds = u^{-\frac{1}{4}}\mu(u).$$

This proves (1.53). It follows that $\lambda(s)$ satisfies (1.52).

6.5. We start now from the formula

$$\frac{\sin xy}{y} = \frac{\sqrt{\pi}}{4\pi i}\int_{c-i\infty}^{c+i\infty} e^{x^2 s} s^{-\frac{1}{4}} e^{-y^2/4s}\,ds, \tag{6.51}$$

valid for all positive x and y.† Hence

$$\frac{1-\cos xy}{y^2} = \frac{\sqrt{\pi}}{4\pi i}\int_{c-i\infty}^{c+i\infty} e(x,s) s^{-\frac{1}{4}} e^{-y^2/4s}\,ds, \tag{6.52}$$

† The formula is a special case of one of the classical integral representations of the Bessel functions: take $\nu = \frac{1}{2}$ in Watson (**30**), 177, formula (8).

where
$$e(x,s)=\int_0^x e^{w^2 s}\,dw. \tag{6.53}$$

It is plain that $e(x, c+it)$ is bounded in any finite t interval; also

$$e(x,c+it)=e^{x^2 c}\int_{x'}^{x} e^{w^2 it}\,dt=O(|t|^{-\frac{1}{2}}) \tag{6.54}$$

for large t.

Multiplying (6.52) by $f(y)$, and integrating,

$$\int_0^\infty \frac{1-\cos xy}{y^2}f(y)\,dy=\frac{\sqrt{\pi}}{4\pi i}\int_0^\infty f(y)\,dy\int_{c-i\infty}^{c+i\infty} e(x,s)s^{-\frac{3}{2}}e^{-y^2/4s}\,ds. \tag{6.55}$$

The integral is absolutely convergent. To prove this, it is plainly sufficient to show that

$$\int_{-\infty}^{\infty}|e(x,c+it)|(1+|t|)^{-\frac{3}{2}}e^{-cy^2/4(c^2+t^2)}\,dt$$

is $O(1)$ for small y and $O(y^{-1})$ for large y. The first result follows at once from the boundedness of $e(x, c+it)$ and the convergence of $\int(1+|t|)^{-\frac{3}{2}}\,dt$. As regards the second, the integral is less than a constant multiple of

$$\int_0^\infty \frac{e^{-cy^2/4(c^2+t^2)}}{(1+t)^2}\,dt<\int_0^c e^{-y^2/8c}\,dt+\int_c^\infty \frac{e^{-cy^2/8t^2}}{t^2}\,dt$$

$$<ce^{-y^2/8c}+\frac{1}{y}\int_0^\infty e^{-\frac{1}{8}cu^2}\,du=O\left(\frac{1}{y}\right).$$

Inverting the integrations in (6.55), we obtain

$$\int_0^\infty \frac{1-\cos xy}{y^2}f(y)\,dy=\frac{\sqrt{\pi}}{4\pi i}\int_{c-i\infty}^{c+i\infty} e(x,s)s^{-\frac{1}{2}}\,ds\int_0^\infty f(y)e^{-y^2/4s}\,dy$$

$$=\frac{\sqrt{(2\pi)}}{4\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e(x,s)}{s}\lambda(2s)\,ds,$$

by (1.52). If we differentiate twice with respect to x, we find

$$\int_0^\infty \cos xy\,f(y)\,dy=\frac{\sqrt{(\frac{1}{2}\pi)}x}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{4}x^2 s}\lambda(s)\,ds=\sqrt{(\tfrac{1}{2}\pi)}f(x).$$

The legitimacy of the differentiations results from the uniform convergence of the final integrals, proved in § 6.3; and this completes the proof of the theorem.

6.6. The corresponding formula for general ν is

$$f(x) = \frac{x^{\frac{1}{2}-\nu}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}sx^2} s^{-\frac{1}{2}(\nu+1)} \mu(s)\, ds.$$

As examples, besides the one mentioned in § 1.4, we have:

$$\mu(s) = 2^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}a)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}a)(s^{\frac{1}{2}a-\frac{1}{2}}+s^{\frac{1}{2}-\frac{1}{2}a}),$$

$$f(x) = 2^{\frac{1}{2}a}\Gamma(\tfrac{1}{2}a)x^{-a}+2^{\frac{1}{2}(1-a)}\Gamma\{\tfrac{1}{2}(1-a)\}x^{-(1-a)},$$

$$\mu(s) = \sqrt{(\tfrac{1}{2}\pi)}\,\frac{1}{\sqrt{(s^{\frac{1}{2}}+s^{-\frac{1}{2}})}}, \qquad f(x) = e^{-\frac{1}{2}x^2},$$

$$\mu(s) = \frac{\Gamma(\frac{1}{4})}{\sqrt{(2\pi)}}\left(s+\frac{1}{s}\right)^{-\frac{1}{2}}, \qquad f(x) = x^{\frac{1}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}x^2)$$

(all for $\nu = -\frac{1}{2}$).

As an example of the general case, take

$$\mu(s) = \exp\left\{-\tfrac{1}{2}b^2\left(s+\frac{1}{s}\right)\right\}.$$

It will be found that we obtain the example (7) of § 3.

6.7. There is an 'L^2' theorem connected with the second solution, but it involves the convergence of

$$\int_0^\infty e^{-cx^2} f^2(x)\,\frac{dx}{x} \quad (c>0)$$

and is not very useful. The theorem corresponding to Theorem **13** is

THEOREM **17**. *If the integrals*

$$\int_0^1 |f|\,\frac{dx}{x}, \qquad \int_1^\infty |f|\,\frac{dx}{x^2}$$

are finite, and (1.12) *holds for all positive x, the integral being defined as a Cauchy integral at infinity, then $f(x)$ is of the form (B), $\mu(s)$ being regular, and satisfying* (1.54), *for $\sigma > 0$.*

The proof is very similar to that of Theorem **13**, but rather simpler, and there is no condition corresponding to (ii) of Theorem **13**.

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AN EXPRESSION FOR THE NUMBER OF LATTICE-POINTS IN A CIRCLE

By S. W. P. STEEN

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THE object of this paper is to obtain for the number of lattice-points in the circle $n^2+m^2 \leqslant x$ (points on the circle being counted one-half times) the expression,

$$A(x) = 16\pi x \sum_{\nu=0}^{\infty} (-)^{\nu}(2\nu+1)\left\{\sum_{n=0}^{\infty}{}' \frac{J_{2\nu+1}(2\pi n\sqrt{x})}{2\pi n\sqrt{x}}\right\}^2$$

where the dash indicates that the term $n=0$ is to be multiplied by $\frac{1}{2}$ and this term is taken to mean the limit as $n \to 0$.

Let $\qquad 0 \leqslant \theta_n \leqslant \frac{1}{2}\pi$, for $0 \leqslant n \leqslant \sqrt{x}$,

so that $\qquad 0 \leqslant \theta_n \pm \theta_m + \frac{1}{2}\pi < 2\pi$.

Further, let

$$\epsilon^{-}_{n,m} = 0, \qquad \epsilon^{+}_{n,m} = \begin{cases} 1 & \text{if } \theta_n+\theta_m > \frac{1}{2}\pi, \\ \frac{1}{2} & \text{if } \theta_n+\theta_m = \frac{1}{2}\pi, \\ 0 & \text{if } \theta_n+\theta_m < \frac{1}{2}\pi. \end{cases}$$

Define θ_n by the equations $\cos\theta_n = n/\sqrt{x}, \qquad \sin\theta_n = \sqrt{(1-n^2/x)}$.

Then $\quad \epsilon^{+}_{n,m} = 1$, when $\cos(\theta_n+\theta_m) < 0$, i.e. when $x > n^2+m^2$;

$\epsilon^{+}_{n,m} = \frac{1}{2}$, when $\cos(\theta_n+\theta_m) = 0$, i.e. when $x = n^2+m^2$;

$\epsilon^{+}_{n,m} = 0$, when $\cos(\theta_n+\theta_m) > 0$, i.e. when $x < n^2+m^2$.

Hence

$$A(x) = 4 \sum_{n,m=0}^{\sqrt{x}}{}' \epsilon^{+}_{n,m} = 4 \sum_{n,m=0}^{\sqrt{x}}{}' \epsilon^{+}_{n,m} - 4 \sum_{n,m=0}^{\sqrt{x}}{}' \epsilon^{-}_{n,m},$$

where the dash indicates that the terms $n=0=m$, $n=0=m-\sqrt{x}$, and $n-\sqrt{x}=0=m$ are to be multiplied by $\frac{1}{4}$, and that the terms $n=0$, $0<m<\sqrt{x}$, and $0<n<\sqrt{x}$, $m=0$ are to be multiplied by $\frac{1}{2}$.

Let

$$P_1(x) = -[x]+x-\tfrac{1}{2} = \frac{1}{\pi}\sum_{\nu=1}^{\infty} \frac{\sin 2\pi x\nu}{\nu}, \qquad P_1(0) = 0,$$

then $\quad \epsilon^{+}_{n,m} = 2\left[P_1\left(\frac{\theta_n+\theta_m+\frac{1}{2}\pi}{2\pi}\right) - \tfrac{1}{2}P_1\left(\frac{\theta_n+\theta_m+\frac{1}{2}\pi}{\pi}\right)\right],$

and similarly

$$\epsilon^{-}_{n,m} = 2\left[P_1\left(\frac{\theta_n - \theta_m + \frac{1}{2}\pi}{2\pi}\right) - \tfrac{1}{2}P_1\left(\frac{\theta_n - \theta_m + \frac{1}{2}\pi}{\pi}\right)\right].$$

Thus

$$\epsilon^{\pm}_{n,m} = \frac{2}{\pi}\left[\sum_{\nu=1}^{\infty}\frac{\sin\nu(\theta_n \pm \theta_m + \frac{1}{2}\pi)}{\nu} - \sum_{\nu=1}^{\infty}\frac{\sin 2\nu(\theta_n \pm \theta_m + \frac{1}{2}\pi)}{2\nu}\right]$$

$$= \frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{\sin(2\nu+1)(\theta_n \pm \theta_m + \frac{1}{2}\pi)}{2\nu+1} = \frac{2}{\pi}\sum_{\nu=0}^{\infty}(-)^{\nu}\frac{\cos(2\nu+1)(\theta_n \pm \theta_m)}{2\nu+1}.$$

Thus

$$A(x) = 4\sum_{n,\,m=0}^{\sqrt{x}}{}' (\epsilon^{+}_{n,m} - \epsilon^{-}_{n,m})$$

$$= \frac{8}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{2\nu+1}\sum_{n=0}^{\sqrt{x}}{}'\sum_{m=0}^{\sqrt{x}}{}'\{\cos(2\nu+1)(\theta_n - \theta_m) - \cos(2\nu+1)(\theta_n + \theta_m)\}$$

$$= \frac{16}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{2\nu+1}\left\{\sum_{n=0}^{\sqrt{x}}{}'\sin(2\nu+1)\theta_n\right\}^2,$$

where the dash on the single summation means that the terms $n = 0$ and $n = \sqrt{x}$, if present, are to be multiplied by $\frac{1}{2}$. The last series, being composed of a finite number of convergent series, is thus convergent.

Now if $n \neq 0$, $\nu > 0$*

$$\frac{J_{2\nu+1}(2\pi n\sqrt{x})}{2\pi n\sqrt{x}} = \frac{n^{2\nu}(2\pi\sqrt{x})^{-2\nu-2}}{2^{2\nu}\Gamma(2\nu+\frac{3}{2})\Gamma(\frac{1}{2})}\int_0^{2\pi\sqrt{x}}(4\pi^2 x - t^2)^{2\nu+\frac{1}{2}}\cos nt\,dt$$

$$= (-)^{\nu}\frac{(2\pi\sqrt{x})^{-2\nu-2}}{2^{2\nu}\Gamma(2\nu+\frac{3}{2})\Gamma(\frac{1}{2})}\int_0^{2\pi\sqrt{x}}\frac{d^{2\nu}}{dt^{2\nu}}\{(4\pi^2 x - t^2)^{2\nu+\frac{1}{2}}\}\cos nt\,dt.$$

If $\nu > 0$ and $n = 0$, and if the left-hand side be interpreted as a limit, then both sides vanish for $n = 0$, so that the result is true for $\nu > 0$ and $n \geqslant 0$. Thus for $\nu > 0$

$$\lim_{N\to\infty}\sum_{n=0}^{N}\frac{J_{2\nu+1}(2\pi n\sqrt{x})}{2\pi n\sqrt{x}}$$

$$= \lim_{N\to\infty}\frac{(-)^{\nu}(2\pi\sqrt{x})^{-2\nu-2}}{2^{2\nu}\Gamma(2\nu+\frac{3}{2})\Gamma(\frac{1}{2})}\int_0^{2\pi\sqrt{x}}\frac{d^{2\nu}}{dt^{2\nu}}\{(4\pi^2 x - t^2)^{2\nu+\frac{1}{2}}\}\left(\frac{\frac{1}{2}\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t} - \frac{1}{2}\right)dt$$

$$= (-)^{\nu}\frac{(2\pi\sqrt{x})^{-2\nu-2}\pi}{2^{2\nu}\Gamma(2\nu+\frac{3}{2})\Gamma(\frac{1}{2})}\sum_{r=0}^{\sqrt{x}}{}'\left[\frac{d^{2\nu}}{dt^{2\nu}}(4\pi^2 x - t^2)^{2\nu+\frac{1}{2}}\right]_{t=2\pi n},$$

* Watson, *Theory of Bessel Functions*, 1922, p. 48.

while for $\nu = 0$

$$\sum_{n=0}^{\infty}{}' \frac{J_1(2\pi n\sqrt{x})}{2\pi n\sqrt{x}} = \frac{(2\pi\sqrt{x})^{-2\pi}}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} \sum_{n=0}^{\sqrt{x}}{}' \sqrt{(4\pi^2 x - 4\pi^2 n^2)}$$

$$= \frac{1}{\pi\sqrt{x}} \sum_{n=0}^{\sqrt{x}}{}' \sin\theta_n.$$

Again*

$$\frac{d^{2\nu}}{dt^{2\nu}}\{(1-t^2)^{2\nu+\frac{1}{2}}\} = \frac{(4\nu+1)!}{2^{2\nu}(2\nu+1)!}\sqrt{(1-t^2)}C^1_{2\nu}(t),$$

where $C^1_{2\nu}(t)$ is the coefficient of $\alpha^{2\nu}$ in the expansion by ascending powers of α of the expression

$$\frac{1}{2\sqrt{(t^2-1)}}\left[\frac{1}{\alpha - t - \sqrt{(t^2-1)}} - \frac{1}{\alpha - t + \sqrt{(t^2-1)}}\right] = \frac{1}{\alpha^2 - 2\alpha t + 1}.$$

Thus

$$C^1_{2\nu}(n/\sqrt{x}) = \frac{\sin(2\nu+1)\theta_n}{\sqrt{(1-n^2/x)}} = \frac{2^{2\nu}(2\nu+1)!}{(4\nu+1)!} \frac{\left[\frac{d^{2\nu}}{dt^{2\nu}}(4\pi^2 x - t^2)^{2\nu+\frac{1}{2}}\right]_{t=2\pi n}}{\sqrt{(1-n^2/x)}(2\pi\sqrt{x})^{2\nu+1}}.$$

Hence, for $\nu \geqslant 0$

$$\sum_{n=0}^{\infty}{}' \frac{J_{2\nu+1}(2\pi n\sqrt{x})}{2\pi n\sqrt{x}} = \frac{(-1)^\nu}{(2\nu+1)\pi\sqrt{x}} \sum_{n=0}^{\sqrt{x}}{}' \sin(2\nu+1)\theta_n,$$

and thus $$A(x) = 16\pi x \sum_{\nu=0}^{\infty} (-)^\nu (2\nu+1)\left\{\sum_{n=0}^{\infty}{}' \frac{J_{2\nu+1}(2\pi n\sqrt{x})}{2\pi n\sqrt{x}}\right\}^2,$$

as was to be proved. It is interesting to note that, if we formally interchange the orders of summation, we obtain the equations

$$A(x) = 16\pi x \sum_{n=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' \sum_{\nu=0}^{\infty} (-)^\nu (2\nu+1) \frac{J_{2\nu+1}(2\pi n\sqrt{x})}{2\pi n\sqrt{x}} \cdot \frac{J_{2\nu+1}(2\pi m\sqrt{x})}{2\pi m\sqrt{x}}$$

$$= 8\pi x \sum_{n=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' \frac{J_1(2\pi\sqrt{\{(n^2+m^2)x\}})}{2\pi\sqrt{\{(n^2+m^2)x\}}}$$

$$= \sqrt{x} \sum_{n=0}^{\infty} U(n) \frac{J_1(2\pi\sqrt{\{nx\}})}{\sqrt{n}},$$

where we have used the addition-theorem for Bessel functions,‡ and where $U(n)$ is the number of solutions of the equation $n = n_1^2 + n_2^2$ in positive or negative integers. The last expression for $A(x)$ is, of

* Whittaker and Watson, *Modern Analysis*, 2nd ed., 1915, p. 323.

‡ Watson, loc. cit., p. 363 equation (2) where $\nu = 1$, $Z = 2\pi n\sqrt{x}$, $z = 2\pi m\sqrt{x}$.

course, well known. The principal term in the series whose sum we have shown to be equal to $A(x)$ is the term $\nu = 0$. This term is equal to

$$\frac{16}{\pi}\Big[\sum_{n=0}^{\sqrt{x}}{}' \sqrt{(1-n^2/x)}\Big]^2 = \pi x + O(\sqrt{x}),$$

for

$$\sum_{n=0}^{\sqrt{x}}{}' \sqrt{(1-n^2/x)} = \int_0^{\sqrt{x}} \sqrt{(1-u^2/x)}\, du + O\left(\int_0^{\sqrt{x}} \frac{u}{x\sqrt{(1-u^2/x)}}\, du\right)$$

$$= \tfrac{1}{4}\pi\sqrt{x} + O(1).$$

A DIRECT PROOF OF NICHOLSON'S INTEGRAL

By A. L. DIXON AND W. L. FERRAR

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In a recent paper* we proved a number of formulae related to Nicholson's well-known result

$$J_\nu^2(x)+Y_\nu^2(x)=\frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh 2\nu t\,dt. \tag{1}$$

In this note we return to the method used by Nicholson† himself and, by combining it with a process used in our previous paper, we show that a short and rigorous proof can be obtained by means of the transformations which led to the discovery of the formula.

We start from the established formulae W. 180‡

$$K_\nu(ix)=-\tfrac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi i}H_\nu^{(2)}(x)=\int_0^\infty e^{-ix\cosh t}\cosh\nu t\,dt,$$

$$K_\nu(-ix)=\tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(x)=\int_0^\infty e^{ix\cosh t}\cosh\nu t\,dt,$$

valid when $x>0$ and $|R(\nu)|<1$, and get at once

$$J_\nu^2(x)+Y_\nu^2(x)=(4/\pi^2)K_\nu(ix)K_\nu(-ix)$$

$$=\frac{1}{\pi^2}\lim_{R\to\infty}\int_{-R}^{R}\int_{-R}^{R}e^{ix(\cosh t-\cosh u)}\cosh\nu t\cosh\nu u\,dudt. \tag{2}$$

If now $|R(\nu)|<\frac{1}{2}$, we can change the variables by means of the substitution $\qquad t+u=2T, \qquad t-u=2U,$

and write§ (2) as

$$\frac{2}{\pi^2}\lim_{R\to\infty}\int_{-R}^{R}\int_{-R}^{R}e^{-2ix\sinh T\sinh U}\cosh 2\nu T\,dTdU$$

$$=\frac{8}{\pi^2}\lim_{R\to\infty}\int_0^R\int_0^R\cos(2x\sinh T\sinh U)\cosh 2\nu T\,dTdU, \tag{3}$$

* *Quart. Journ. of Math.* (Oxford), 1 (1930), 122–45.

† *Phil. Mag.* (6), 19 (1910), 234; *Quart. Journ. of Math.* 42 (1911), 221.

‡ W. 180 means Watson, *Theory of Bessel Functions*, p. 180, and similarly for other references to the same book.

§ Integration by parts of $\displaystyle\int\frac{i\cosh\nu u}{x\sinh u}\,d(e^{-ix\cosh u})$ is the crux of the proof

or, on changing the order of integration in the finite integral,

$$\frac{8}{\pi^2}\lim_{R\to\infty}\int_0^R\int_0^R \cos(2x\sinh T\sinh U)\cosh 2\nu T\,dU dT. \tag{4}$$

But since $|R(\nu)|<\frac{1}{2}$, we can show that (3) is equal to

$$\frac{8}{\pi^2}\lim_{R\to\infty}\int_0^R\int_0^\infty \cos(2x\sinh T\sinh U)\cosh 2\nu T\,dT dU$$

$$=\frac{8\cos\nu\pi}{\pi^2}\int_0^\infty K_{2\nu}(2x\sinh U)\,dU, \tag{5}$$

on using the formula W. 183 (13) for $K_{2\nu}(z)$.

Taking the form (4), we can show that it is equal to

$$\frac{8}{\pi^2}\lim_{R\to\infty}\int_0^R\int_0^\infty \cos(2x\sinh T\sinh U)\cosh 2\nu T\,dU dT$$

$$=\frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh T)\cosh 2\nu T\,dT, \tag{6}$$

on using W. 183 (13) with $\nu=0$.

By the theory of analytic continuation we see, from (6), that

$$J_\nu^2(z)+Y_\nu^2(z)=\frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t\,dt \tag{7}$$

is valid for all values of ν if $R(z)>0$, while, from the continuity of the integral we see, further, that (7) is also true when z is a pure imaginary* (not zero) provided that $|R(\nu)|<\frac{3}{4}$.

that the various steps are legitimate. Further details are given in § 4.11 of our previous paper.

* If $|R(\nu)|<\frac{3}{4}$, and $z=x+iy$, then, for $y\neq 0$, each side of (7) with $x=0$ is the limit of that side as $x\to 0$ through positive values. The asymptotic value of $K_0(2z\sinh t)$ and integration by parts are needed to deal with the right-hand side.

Similarly, (5) shows that

$$J_\nu^2(z)+Y_\nu^2(z) = \frac{8\cos\nu\pi}{\pi^2}\int_0^\infty K_{2\nu}(2z\sinh t)\,dt, \tag{8}$$

the formula being valid when $|R(\nu)| < \frac{1}{2}$ and $R(z) \geqslant 0$ (the particular value $z = 0$ excluded). The restriction on ν is necessary for convergence at the lower limit of integration.